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ASW SONAR
TECHNOLOGY REPORT

A UNIFIED THEORY FOR OPTIMUM ARRAY PROCESSING

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DEPARTMENT OF THE NAVY
NAVAL SHIP SYSTEMS COMMAND

NObsr-93055
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Task 11353

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A UNIFIED THEORY FOR OPTIMUM ARRAY PROCESSING

by

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I. INTRODUCTION

In this report, we consider in some detail various approaches to the problem of processing data from arrays. This problem, which is frequently referred to as the combined space-time processing problem, will be discussed in the context of a sonar system. The general results are applicable to array processing in any area.

In Section A, we will discuss some of the considerations that arise in modelling the sonar problem and indicate some of the implications of the various models. In Section B, we develop in detail the quantitative model that we will use. In Section C, we outline the organization of the report and summarize the principal results.

Before proceeding, it is appropriate to mention the background assumed of the reader. A knowledge of random process theory at the level of Davenport and Root[1] is necessary. In addition, the elements of statistical detection theory (e.g., Helstrom[2] or Ref. 1, Chap. 14) and linear filtering theory (Ref. 1, Chap. 11) are needed. Matrix notation and a few simple matrix properties are used (e.g., Hildebrand[3] or Bellman[4]). In various portions, additional background is needed; this background is contained in Van Trees[5].

A. PRELIMINARY CONSIDERATIONS

The basic system of interest is shown in Figure 1. The waveforms are received by an array of hydrophones. These waveforms contain a component due to various noise sources, which may be within the hydrophones or external to them. In the case of active sonars, there is also reverberation return. If a target is present, a "signal" component will be added. In the active sonar case, this signal is a reflection of the transmitted signal from the target. In the passive sonar case, the signal consists of sound generated by the target itself. The purpose of processing is to obtain information about the target from the received waveforms.

The first problem is to develop a suitable model for the signals and noises.

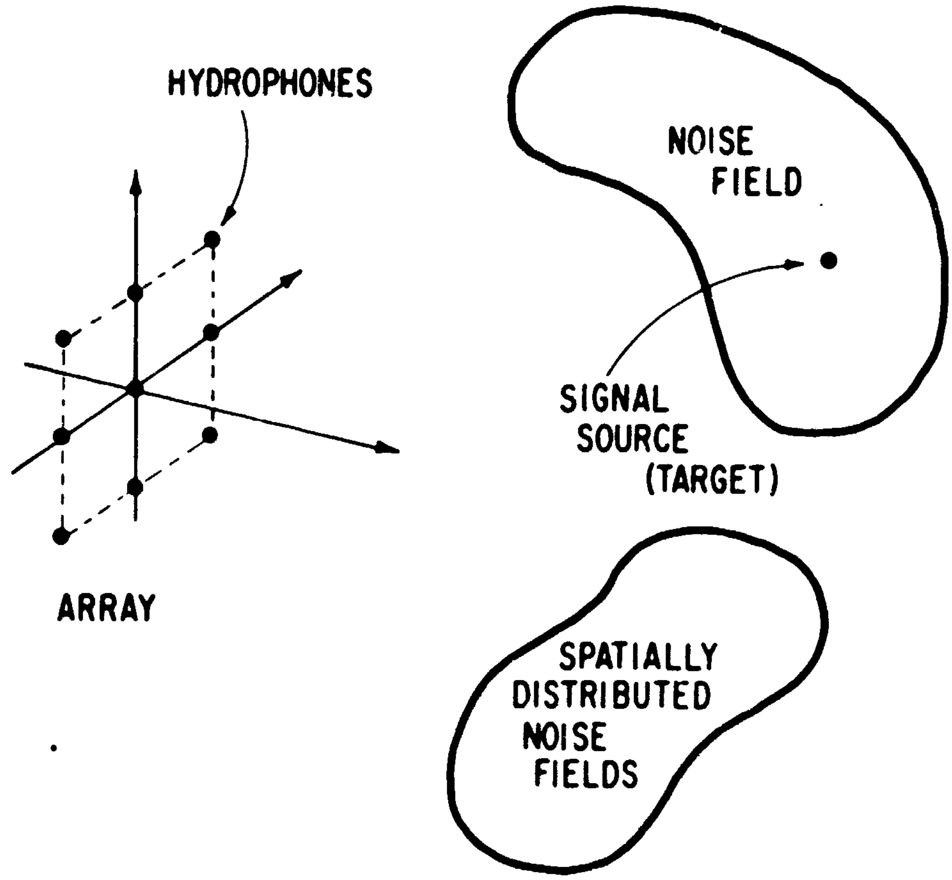

FIGURE 1   SPACE-TIME PROCESSING MODEL

The received waveforms correspond to the N hydrophone outputs: $r_1(t), r_2(t), \cdots r_N(t)$. It is convenient to describe these as a single vector signal $\underline{r}(t)$.

$$\underline{r}(t) \triangleq \begin{bmatrix} r_1(t) \\ r_2(t) \\ \cdot \\ \cdot \\ \cdot \\ r_N(t) \end{bmatrix} \qquad T_i \leq t \leq T_f$$

(T_i is the initial observation time, T_f is the final observation time.)

Similarly, we can describe the signal component in the N waveforms as $\underline{s}(t)$ and the noise component in the N waveforms as $\underline{n}(t)$. Thus, when a target is present,

$$\underline{r}(t) = \underline{s}(t) + \underline{n}(t) \quad T_i \le t \le T_f$$

First, we consider possible methods of characterizing the signal component $\underline{s}(t)$. Four possible characterizations come to mind:

1. Deterministic (Known) Signals - In an active system, the shape of the transmitted signal is known. If the channel does not distort the signal and the target acts as a perfect point reflector, then the shape of the signal can be considered known. This particular model is rarely appropriate for the sonar case.

2. Deterministic Signals with Unknown Parameters - Normally the transmitted signal is centered around center-frequency. For example, the signal illuminating the target might be

$$s_I(t) = \sin \omega_c t$$

Even a simple target will introduce a phase angle and an attenuation which will be unknown to the receiver. Thus, the reflected signal might be

$$s_R(t) = V_R \sin (\omega_c t + \theta_R)$$

where V_R and θ_R are unknown. Frequently, it is reasonable to assume that θ_R is a random variable with a uniform probability density over the interval $[0, 2\pi]$. The constant V_R can be modelled as either a random variable or as an unknown, non-random variable. For a simple non-dispersive channel which is essentially constant during the signalling interval, the constants can include its attenuation and phase shift

3. Random Signals - If the channel or target fluctuates while being illuminated by the signal, then the signal will be distorted. Since these fluctuations are inherently random, a convenient approach is to view the signal as a sample function from a random process.

In the passive sonar case, the signal emitted from the target is caused by a variety of things such as engine noise or propeller noise. Here, the signal source is inherently random.

In both of these cases, one usually knows enough about the physical situation to be able to specify the approximate second-moment characteristics of the process (i.e., the mean value function and the covariance function). Frequently the physical origins of the randomness are such that one can assume the process is a Gaussian random process. In this case, the second-moment properties provide a complete characterization.

Once we have assigned statistical properties to the signal, we would expect that any optimum processing schemes we design will be good on the average. In other words, any particular time the experiment is conducted the performance may be good or bad. but when averaged over the assumed signal ensemble the performance will be optimum.

In many cases it is difficult to assign statistical properties to the signal. Alternately, we may want to design a test that will be optimum for the particular signal present (as contrasted to a signal ensemble). In these situations, a fourth model is appropriate.

4. Non-Random, Unknown Signals - In this case, we assume that the N signal inputs $s_1(t)$, $s_2(t)$, $\cdots$ $s_N(t)$ are unknown. We want to design processors that are good for the particular set that is present (not some average signal set).

Next, we consider characterizations for the noise. As mentioned earlier, the possible sources include 1) hydrophone noise, 2) ambient noise, and 3) reverberation noise in active systems. The first two arise from the combined effect of many small, independent sources. According to the Central Limit Theorem, one can model these as Gaussian (normally stationary) random processes.

Reverberation noise is caused by the reflection of the transmitted signal from various objects in the ocean. Its properties will be a function of the transmitted signal shape and the reflection mechanism. A detailed derivation of a possible mathematical model is given by Kelly and Lerner[(6)] (see also Refs. 7 and 8). If one assumes a large number of small reflectors, one is led to a Gaussian process model which is non-stationary and has characteristics that depend on the transmitted signal.

Accordingly, we will assume that the noise is a sample function from a Gaussian random process with known statistical properties.

In addition to developing a suitable model for the signal and noise environment, we must establish the goal of our processing system and specify a criterion to measure how closely the system achieves this goal.

We will discuss some possible criteria for the various signal models suggested above.

1. Case 1

When the shape of the transmitted signal is known, the question of interest is normally whether or not the target is present. This is a binary hypothesis testing problem. The received waveform under the two hypotheses is:

$$H_1 : \underline{r}(t) = \underline{s}(t) + \underline{n}(t) \qquad T_i \leq t \leq T_f \text{ (target present)}$$

$$H_0 : \underline{r}(t) = \qquad \underline{n}(t) \qquad T_i \leq t \leq T_f \text{ (target absent)}$$

Using either a Bayes or Neyman-Pearson criterion, one is led to a likelihood ratio test. One operates on the waveform $\underline{r}(t)$ to construct the function,

$$\Lambda(\underline{r}(t)) \triangleq \frac{\text{"}p(\underline{r}(t)|H_1)\text{"}}{\text{"}p(\underline{r}(t)|H_0)\text{"}}$$

(The quotation marks emphasize that we must be careful about the meaning of these expressions.)

One then compares Λ to some threshold η, which is chosen as a function of cost in the Bayes case or to achieve a desired P_F in the Neyman-Pearson case.

The resulting processor for known $\underline{s}(t)$ turns out to be a matrix linear filter with N inputs and 1 output.

Alternately, one cannot assume the noise is Gaussian if only the second-moment properties of the noise are known. Then, one does not have enough information to construct $\Lambda(\underline{r}(t))$. A plausible approach in this case is to require a linear processor (once again, with N inputs and 1 output) and try to maximize the output signal-to-noise ratio,

$$\frac{S}{N} \triangleq \frac{(\text{output due to } \underline{s}(t) \text{ at } T_f)^2}{E\left[(\text{output due to } \underline{n}(t) \text{ at } T_f)^2\right]}$$

(The symbol E denotes expectation.)

It is straightforward to show that these two criteria lead to identical results.

2. Case 2

If the unknown parameters are random and the problem is one of detection, the procedure is identical to Case 1. However, if the parameters are non-random or if we want to estimate them, the procedure must be modified.

3. Case 3

For random signals, two possibilities exist.

First, if detection is the problem of interest, we are once again led to a likelihood ratio test. To construct the likelihood function, it is necessary to characterize the signal process completely. The most common process model is a Gaussian model. As discussed above, the second-moment characterization then provides a complete description. In our detailed discussion, we will encounter certain cases in which the Gaussian signal assumption will be valid. The resulting processor is a quadratic device.

Second, it is often desirable to estimate what the signal component of the input is. Since both the signal and noise are sample functions of a random process, a minimum mean-square error estimate is appropriate. We denote this MMSE estimate by $\hat{\underline{s}}(t)$. Under the assumption that the signal and noise process are both Gaussian, $\hat{\underline{s}}(t)$ is obtained by using a linear processor. On the other hand, the Gaussian signal process assumption is not invoked, the form of the processor must be specified. If we ask for the best linear MMSE estimate, we can solve the problem using only second-moment properties of the process. The resulting estimates are the same in both cases.

4. Case 4

For non-random but unknown signals, the desired procedure is less obvious. There are two possibilities, which are analogous to those outlined in Case 3.

One can construct a generalized likelihood ratio test:

$$\Lambda_g(\underline{r}(t)) \triangleq \frac{\max\limits_{\underline{s}(t)} \text{"}p[\underline{r}(t) \mid H_1]\text{"}}{\text{"}p[\underline{r}(t) \mid H_0]\text{"}}$$

and compare Λ_g to some threshold adjusted to give the desired false alarm probability. The numerator is found by making a maximum likelihood estimate of $\underline{s}(t)$ and substituting it into the probability density. This test has no claim to optimality but is intuitively logical and frequently performs well.

The analogy to a minimum mean-square error estimate of a sample function is the maximum likelihood estimate of a non-random function. If one does not invoke the Gaussian assumption on the noise, then it is appropriate to ask what sort of estimate of $\underline{s}(t)$ can be obtained using a linear filter. An appropriate criterion might be to require the output to equal $\underline{s}(t)$ exactly <u>if</u> the noise were absent. Subject to this constraint, we design the linear filter to minimize the distortion due to noise. The resulting estimate is found to be the same as the maximum-likelihood estimate.

In this section, we have discussed some of the considerations that are involved in choosing an appropriate mathematical model for the physical problem of interest. We will concentrate our attention on Cases 3 and 4; the noise will be modelled as a sample function from a Gaussian random process, and the signal will be modelled either as a sample function from a Gaussian random process <u>or</u> as a non-random, but unknown, waveform.

We shall now specify this model in detail.

B. QUANTITATIVE MODEL

The general model is easily stated. The received waveforms of interest are $r_1(t), \cdots r_N(t)$. First, consider the noise components, $n_1(t), \cdots n_N(t)$. We denote these by the vector $\underline{n}(t)$. The following properties are assumed:

1. Each noise $n_i(t)$; $i = 1, 2, \cdots N$ is a sample function from a <u>zero-mean</u> Gaussian process.

2. The noise processes are jointly Gaussian.

3. Statements (1) and (2) are equivalent to saying that $\underline{n}(t)$ is a zero-mean vector Gaussian process.

4. The vector process is completely characterized by its covariance matrix,

$$\underline{K}_n(t,u) \triangleq E\left[\underline{n}(t)\,\underline{n}^T(u)\right] = \begin{bmatrix} K_{n_1 n_1}(t,u) & K_{n_1 n_2}(t,u) & \cdots & K_{n_1 n_N}(t,u) \\ K_{n_2 n_1}(t,u) & K_{n_2 n_2}(t,u) & & \vdots \\ \vdots & & \ddots & \\ K_{n_N n_1}(t,u) & & & K_{n_N n_N}(t,u) \end{bmatrix}$$

(Observe that the covariance function and correlation function are the same because of the zero-mean assumption.)

5. We further assume that each noise function contains a non-zero "white" component which is independent of the remaining noise function and of the "white" components in the other noise waveforms. Thus, we may write

$$n_1(t) = w_1(t) + n_{c_1}(t)$$

where $w_1(t)$ represents the white component and $n_{c_1}(t)$ is the remaining colored noise component.

$$K_{n_1}(t,u) = K_{w_1}(t,u) + K_{n_{c_1}}(t,u)$$

$$= \frac{N_0}{2}\,\delta(t-u) + K_{n_{c_1}}(t,u)$$

For algebraic simplicity, we assume that the white components in each process are equal. (The general case is a trivial modification.)

The covariance matrix of the vector process is:

$$\underline{K}_n(t,u) = \frac{N_0}{2} \delta(t-u)\,\underline{I} + \underline{K}_c(t,u)$$

where $\underline{I}$ is the identity matrix and $\underline{K}_c(t,u)$ is the covariance matrix of the colored noises.

Physically, this white component may correspond to internal noise in the hydrophone and its associated circuitry. Since its bandwidth is wider than any signals of interest, we may model it as a process with a flat spectrum at all frequencies.

6. We assume that the colored noise component has finite energy. This implies that

$$\iint_{T_i}^{T_f} \underline{K}_c(t,u)\, \underline{K}_c^T(t,u)\, dt\, du < \infty$$

Observe that we do not need to be explicit about the noise sources. Any Gaussian noise source which leads to the same vector covariance function will be treated alike. Later, in some examples, we will see how various noise fields give rise to particular covariance functions.

The signal is also characterized by a vector $\underline{s}(t)$.

Under the Gaussian assumption, we denote its covariance by $\underline{K}_s(t,u)$ and assume it is zero-mean. It does not contain a white component. For simplicity, we assume it is independent of the noise process.

Observe that this would not be true if the channel or target fluctuated while being illuminated by the signal, because the transmitted signal would influence both the returned signal process and the reverberation noise. The modification to include this coupling is straightforward.

For the general developments, no further restrictions on the components of $\underline{s}(t)$ are needed. A special case that we will emphasize in most of our examples is one in which each signal $s_i(t)$ is a shifted version of the same signal.

$$s_i(t) = s(t - \tau_i) \quad i = 1, 2, \cdots N \tag{1}$$

where the τ_i are known.

Physically, the simplest case this might correspond to is a point target and plane-wave propagation. Neither of these assumptions are necessary.

Under the non-random signal assumption, $\underline{s}(t)$ is simply an unknown vector. Once again. we will emphasize the case described by Eq. (1).

C. ORGANIZATION AND PRINCIPAL RESULTS

In Chapters II and III we study the random signal case in some detail. First, we find the form of the optimum processor for the general case. Then we look at some interesting special cases and evaluate their performance.

In Chapter IV we study the non-random, but unknown, signal problem.

In Chapter V we look at the special case in which the signal vector satisfied Eq. (1)and the noise is homogeneous (i.e., $K_{n_i}(t,u)$ is not a function of i). The notion of array gain is encountered and its significance discussed.

In Chapters VI and VII we consider some particular examples of distributed and directional noise fields.

Finally, in Chapter VIII we summarize the results and suggest some future work.

The principal result is the demonstration that, for a large class of problems, the basic structure of the optimum processor does not depend on the signal case of interest nor on the criterion used. Specifically, we will arrive at a receiver in which the only matrix operation is invariant to the above assumptions and the solutions to all of the above problems appear at various points of the receiver.

A secondary result is some insight into situations in which optimum processing may be worthwhile.

II. GAUSSIAN SIGNALS IN GAUSSIAN NOISE

In this chapter we derive the optimum detector for the random signal process described in Case 3.

The basic derivation is not restricted to the array processing case. Historically, the essential results were first obtained by Price in 1954 during his studies on multiple scatter links for communication.(9, 10, 11) In 1959 the problem was studied in a different context by Wolf.(12) Recently, Bryn(13) re-derived the results for the special case of stationary processes and infinite time intervals. Middleton and Groginsky(14) have also studied the array problem.

A. DERIVATION OF LRT

As discussed in the introduction, the target or channel changes appreciably during the observation interval in many active sonar situations. Thus, even though the transmitted signal is completely known, the effect of the transmission path or the target reflection mechanism causes the returned waveform to have a random behavior. With passive sonar, the actual generated waveform has a non-deterministic character.

A Gaussian model is suitable for this situation. In the absence of a target, the received waveform will consist of contributions due to the various types of noises discussed above. As before, we call this hypothesis H_0:

$$\underline{r}(t) = \underline{n}(t) \qquad [H_0] \tag{2}$$

We will assume that there is a non-zero amount of noise generated at each hydrophone. This noise is statistically independent at the various elements in the array and is assumed to be "white" over the frequency ranges of interest. To emphasize this white noise component, we will occasionally write

$$\underline{n}(t) = \underline{w}(t) + \underline{n}_c(t) \tag{3}$$

The covariance matrix under H_0 is:

$$\underline{K}_0(t,u) \triangleq E\left[\underline{r}(t)\,\underline{r}^T(u) : H_0 \text{ is true}\right]$$

$$= E\left[\underline{n}(t)\,\underline{n}^T(u)\right] \triangleq \underline{K}_n(t,u) \tag{4}$$

$$\triangleq \frac{N_0}{2}\,\underline{I} + \underline{K}_{c_0}(t,u)$$

Under hypothesis H_1, there is an additional random component due to the target:

$$\underline{r}(t) = \underline{s}(t:\Omega) + \underline{n}(t) \tag{5}$$

The signal $\underline{s}(t:\Omega)$ is a sample function from a Gaussian random process. For our present purposes, it is adequate to assume that the process is zero-mean.

The covariance under H_1 is:

$$\underline{K}_1(t,u) \triangleq E\left[\left[\underline{r}(t) - \underline{m}_1(t)\right]\left[\underline{r}^T(u) - \underline{m}_1^T(u)\right]\right] \tag{6}$$

$$\triangleq \underline{K}_s(t,u) + \underline{K}_n(t,u),$$

since we assumed the mean is zero.

To solve the detection problem, we must compute the likelihood ratio and compare it to a threshold. The likelihood ratio test is

$$\Lambda(\underline{r}(t)) = \frac{p\left[\underline{r}(t) : H_1\right]}{p\left[\underline{r}(t) : H_0\right]} \gtrless \tau. \tag{7a}$$

There are several ways of formulating this likelihood ratio. A method that leads to simple interpretation of the result is to construct $\Lambda(\underline{r}(t))$ as a ratio of two fractions:

$$\Lambda(\underline{r}(t)) = \frac{p\left[\underline{r}(t) \mid H_1\right]}{p\left[\underline{r}(t) \mid \underline{w}(t) \text{ only}\right]} \div \frac{p\left[\underline{r}(t) \mid H_0\right]}{p\left[\underline{r}(t) \mid \underline{w}(t) \text{ only}\right]} \tag{7b}$$

It is easy to show that both terms exist. Since they have a similar structure, it is adequate to study the first term in detail.

$$\frac{p\left[\underline{r}(t) \mid H_1\right]}{p\left[\underline{r}(t) \mid \underline{w}(t) \text{ only}\right]} = \lim_{N \to \infty} \frac{\dfrac{1}{(2\pi)^{N/2}} \displaystyle\prod_{i=1}^{N} \lambda_i^{1} \exp\left(-\frac{1}{2} \iint_{T_i}^{T_f} dt\, du\, \underline{r}^T(t) \underline{Q}_1(t,u)\, \underline{r}(u)\right)}{\dfrac{1}{(2\pi)^{N/2}} \displaystyle\prod_{i=1}^{N} \lambda_i^{w} \exp\left(-\frac{1}{N_0} \int_{T_i}^{T_f} \underline{r}^T(t)\, \underline{r}(t)\, dt\right)} \tag{8}$$

where the λ_i are the scalar eigenvalues in the vector Karhunen-Loéve expansion† and $\underline{Q}_1(t,u)$ is the inverse matrix kernel.

The inverse matrix kernel satisfies the equation

$$\int_{T_i}^{T_f} \underline{K}_1(t,u)\, \underline{Q}_1(u,z)\, du = \delta(t-z)\, \underline{I} \tag{9}$$

In terms of the vector eigenfunctions

$$\underline{K}_1(t,u) = \sum_{i=1}^{\infty} \left(\frac{N_0}{2} + \lambda_i^{c1}\right) \underline{\phi}_i(t)\, \underline{\phi}_i^T(u) \tag{10}$$

or

$$\lambda_i^{1} = \frac{N_0}{2} + \lambda_i^{c1} \tag{11}$$

† The vector Karhunen-Loéve expansion was developed by Kelly and Root.[15]

Taking the logarithm of Eq. (8), we have:

$$\ln \Lambda(\mathbf{r}(t)) = + \frac{1}{2} \sum_{i=1}^{\infty} \ln \left(1 + \frac{2}{N_0} \lambda_i^{c_1}\right) - \frac{1}{2} \iint_{T_i}^{T_f} dt\, du\, \underline{r}^T(t) \left[\underline{Q}_1(t,u) - \frac{2}{N_0} \underline{I}\, \delta(t-u)\right] \underline{r}(u) \tag{12a}$$

The first term represents a bias which is not dependent on the received waveform; the second term represents the operation on the received waveform.

We will first develop two canonic forms for the operation on the received data and then develop a convenient expression for the bias term. We will denote the period term by the symbol L. The total test will consist of two terms like Eq. (12a).

Using Eqs. (7b) and (12a), we see that the likelihood ratio test becomes:

$$- \frac{1}{2} \iint_{T_i}^{T_f} dt\, du\, \underline{r}^T(t) \left[\underline{Q}_1(t,u) - \frac{2}{N_0} \underline{I}\, \delta(t-u)\right] \underline{r}(u)$$

$$+ \frac{1}{2} \iint_{T_i}^{T_f} dt\, du\, \underline{r}^T(t) \left[\underline{Q}_0(t,u) - \frac{2}{N_0} \underline{I}\, \delta(t-u)\right] \underline{r}(u) \tag{12b}$$

$$\gtrless \ln \eta - \frac{1}{2} \sum_{i=1}^{\infty} \ln \left(1 + \frac{2}{N_0} \lambda_i^{c_1}\right) + \frac{1}{2} \sum_{i=1}^{\infty} \ln \left(1 + \frac{2}{N_0} \lambda_i^{c_0}\right)$$

$$\triangleq \gamma$$

We will initially investigate the first term on the left-hand side of Eq. (12b).

1. Receiver Structure #1

To find a convenient form we divide the covariance function into two parts:

$$\underline{K}_1(t,u) = \frac{N_0}{2} \delta(t-u)\, \underline{I} + \underline{K}_{c_1}(t,u) \tag{13}$$

We want to find a solution to Eq. (9) with the following form:

$$\underline{Q}_1(t,u) = \frac{2}{N_0}\left\{ \delta(t-u)\,\underline{I} - \underline{h}_{c_1}(t,u) \right\} \tag{14}$$

Substituting into Eq. (9), we have

$$\underline{K}_{c_1}(t,u) = + \frac{N_0}{2}\,\underline{h}_{c_1}(t,u) + \int_{T_i}^{T_f} \underline{K}_{c_1}(t,z)\,\underline{h}_{c_1}(z,u)\,dz \qquad T_i \le t,u \le T_f \tag{15}$$

A formal solution can be obtained by writing

$$\underline{h}_{c_1}(t,u) = \sum_{i=1}^{\infty} h_i^1\,\underline{\phi}_i(t)\,\underline{\phi}_i^T(u) \tag{16}$$

Substituting into Eq. (15) and using Eq. (10), we have:

$$h_i^1 = \frac{\frac{2}{N_o}\lambda_i^{c_1}}{1 + \frac{2}{N_o}\lambda_i^{c_1}} \tag{17}$$

Using Eqs. (12) and (14), we see that the first quadratic form on the left side of Eq. (12b) is:

$$+ \frac{1}{N_0}\iint_{T_i}^{T_f} dt\,du\;\underline{r}^T(t)\,\underline{h}_{c_1}(t,u)\,\underline{r}(u) \tag{18}$$

Now the inner integral in Eq. (18) is familiar from optimum filter theory (e.g., Ref. 5, Chap. 6).

$$\int_{T_i}^{T_f} \underline{h}_{c_1}(t,u)\,\underline{r}(u)\,du = \hat{\underline{s}}(t:\hat{a}:H_1) + \hat{\underline{n}}_c(t:H_1) \tag{19}$$

Thus, the operation represented by the inner integral in Eq. (18) is equivalent to an estimation of the non-white portion of the input, assuming that H_1 is true.

Proceeding in an identical manner with the second term in Eq. (12b), we obtain the receiver shown in Figure 2.

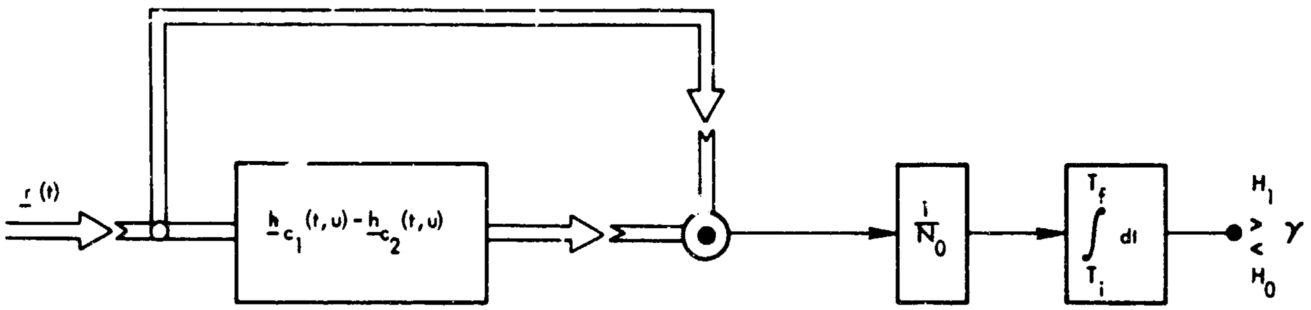

FIGURE 2 RECEIVER STRUCTURE #1

We observe that there is only one filter in the receiver.

$$\underline{h}_{\Delta}(t,u) \triangleq \frac{2}{N_0}\left[\underline{h}_{c_1}(t,u) - \underline{h}_{c_0}(t,u)\right] \tag{20}$$

One can also show that this filter satisfies the following integral equation (see App. A for proof):

$$+\ \underline{K}_s(t,x) = \iint_{T_i}^{Tf} \underline{K}_1(t,u)\,\underline{h}_{\Delta}(u,z)\,\underline{K}_0(z,x)\,du dz \qquad T_i \le t,\ x \le T_f \tag{21}$$

or, using Eq. (6),

$$\underline{K}_s(t,x) = \iint_{T_i}^{T_f} \left[\underline{K}_s(t,u) + \underline{K}_n(t,u)\right]\underline{h}_{\Delta}(u,z)\left[\underline{K}_n(z,x)\right] du\, dz \tag{22}$$

2. Receiver Structure #2

A second interesting interpretation of the optimum receiver can be obtained. We divide $\underline{h}_{\Delta}(u,z)$ into two parts:

$$\underline{h}_{\Delta}(u,z) = \int_{T_i}^{T_f} \underline{g}(u,w)\, \underline{K}_n^{-1}(w,z)\, dw \tag{23a}$$

where

$$\int_{T_i}^{T_f} \underline{K}_n(t,u)\, \underline{K}_n^{-1}(u,z)\, du = \delta(t-z)\underline{I} \tag{23b}$$

Then, substituting into Eq. (22), we obtain an integral equation that $\underline{g}(u,z)$ must satisfy:

$$\underline{K}_s(t,x) = \int_{T_i}^{T_f} \left[\underline{K}_s(t,u) + \underline{K}_n(t,u)\right] \underline{g}(u,x)\, du \qquad T_i \le t,x \le T_f \tag{24}$$

This equation is familiar; the matrix $\underline{g}(u,x)$ is related to the matrix filter one uses to find the minimum mean-square estimate of $\underline{s}(t)$, given the input $\underline{r}(t)$. Specifically,

$$\hat{\underline{s}}(u) = \int_{T_i}^{T_f} \underline{g}^T(x,u)\, \underline{r}(x)\, dx \tag{25}$$

The test statistic is:

$$L = \iint_{T_i}^{T_f} dt\, du\, \underline{r}^T(t)\, \underline{h}_{\Delta}(t,u)\, \underline{r}(u) \tag{26}$$

(The factor of 1/2 is absorbed in the threshold.)

Substituting Eq.(23) into Eq. (26), we have:

$$L = \iiint_{T_i}^{T_f} dt\, du\, dv\; \underline{r}^T(t)\; \underline{g}(t,v)\; \underline{K}_n^{-1}(v,u)\; \underline{r}(u) \tag{27}$$

Writing

$$\underline{z}(v) = \int_{T_i}^{T_f} \underline{K}_n^{-1}(v,u)\; \underline{r}(u)\, du \tag{28}$$

we have

$$L = \int_{T_i}^{T_f} \hat{\underline{s}}^T(v)\; \underline{z}(v)\, dv \tag{29}$$

The receiver shown in Figure 3† is an alternate version of the estimator-correlator of Figure 2 that we find useful in the sequel.

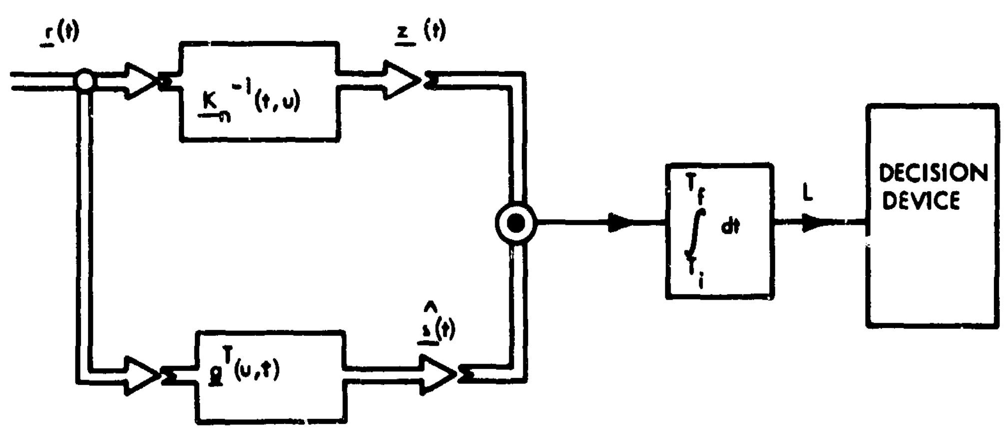

FIGURE 3 ESTIMATOR CORRELATOR RECEIVER: FORM #2 OF THE OPTIMUM RECEIVER

† The symbol ⊙ denotes a dot product $\underline{A}\,\underline{B}^T$. The double lines indicate vector signals.

The vector $\underline{z}(t)$ also has a familiar interpretation from the active sonar case (e.g., Van Trees, Ref. 16 or 5, Chap. 4). It is the input to the correlator, as shown in Figure 4. The other input is $\underline{f}(t)$, the known signal.

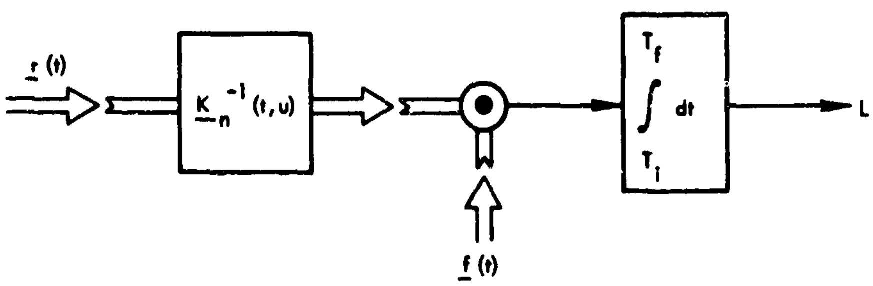

FIGURE 4 ACTIVE SONAR RECEIVER

We now derive a third receiver structure, one that is commonly used in practice.

3. Receiver Structure #3

A third interpretation of the receiver structure is also useful. To obtain this interpretation, we write $\underline{h}_{\Delta}(t,u)$ in terms of an integral.

$$\underline{h}_{\Delta}(t,u) = \int_{T_i}^{T_f} \underline{k}(z,t)\, \underline{k}^T(z,u)\, du \tag{30}$$

Then

$$L = \int_{T_i}^{T_f} dz \int_{T_i}^{T_f} \underline{r}^T(t)\, \underline{k}(z,t)\, dt \int_{T_i}^{T_f} \underline{k}^T(z,u)\, \underline{r}(u)\, du \qquad (31)$$

If we define

$$\underline{x}(z) \triangleq \int_{T_i}^{T_f} \underline{k}^T(z,u)\, \underline{r}(u)\, du\,, \qquad (32)$$

then the two inner integrals correspond to the squared magnitude of $\underline{x}(z)$.

$$L = \int_{T_i}^{T_f} dz \left[\left| \int_{T_i}^{T_f} \underline{r}^T(t)\, \underline{k}(t,z)\, dt \right|^2 \right] \qquad (33)$$

The resulting receiver is shown in Figure 5.

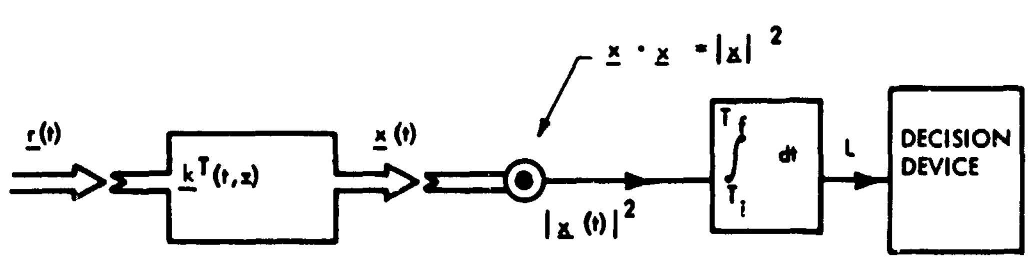

FIGURE 5 FILTER-SQUARER RECEIVER:
FORM #3 OF THE OPTIMUM RECEIVER

The two remaining problems with the receiver are:

1. A convenient closed-form expression for the infinite series on the right hand side of Eq. (12b) that is part of the bias. (This term is evaluated in Appendix B.)

2. Some measure of performance of the receiver. This problem is discussed in the next section.

B. PERFORMANCE OF OPTIMUM RECEIVER

The performance is difficult to compute in the general case. A quantity that provides a reasonably good indication of performance, particularly when the input signal-to-noise level is low, is:

$$d^2 = \frac{\left\{E\left[L \mid H_1\right] - E\left[L \mid H_0\right]\right\}^2}{\mathrm{Var}\left[L \mid H_0\right]} \tag{34}$$

This corresponds to an output signal-to-noise ratio. Its value as a performance indicator has been discussed in detail by Price.[17]

The calculation is straightforward:

$$L = \iint_{T_i}^{T_f} \underline{r}^T(t)\, \underline{h}_\Delta(t,u)\, \underline{r}(u)\, dt\, du \tag{35}$$

$$E\left[L \mid H_1\right] = E_1 \left\{ \iint_{T_i}^{T_f} \underline{r}^T(t)\, \underline{h}_\Delta(t,u)\, \underline{r}(u)\, dt\, du \right\} \tag{36}$$

But

$$\underline{A}^T\, \underline{B}\, \underline{A} = \mathrm{Tr}\, [\, \underline{B}\, (\underline{A}\, \underline{A}^T)\,] \tag{37}$$

So

$$E\left[L \mid H_1\right] = \mathrm{Tr} \iint_{T_i}^{T_f} \underline{h}_\Delta(t,u)\, K_1\, (t,u)\, dt\, du \tag{38}$$

Similarly,

$$E\left[L \mid H_0\right] = \mathrm{Tr} \iint_{T_i}^{T_f} \underline{h}_\Delta(t,u)\, \underline{K}_0(t,u)\, dt\, du \tag{39}$$

Thus, the term in the numerator of Eq. (34) is:

$$\Delta_n \triangleq E\left[L \mid H_1\right] - E\left[L \mid H_0\right] \tag{40}$$

$$\Delta_n = \mathrm{Tr} \iint_{T_i}^{T_f} \underline{h}_\Delta(t,u)\, \underline{K}_s(t,u)\, dt\, du \tag{41}$$

The mean-square value follows in the same manner:

$$E\left[L^2 \mid H_0\right] = E_0 \int_{T_i}^{T_f} \cdot \cdot \int_{T_i}^{T_f} \underline{r}^T(t)\, \underline{h}_\Delta(t,u)\, \underline{r}(u)\, \underline{r}^T(x)\, \underline{h}_\Delta(x,y)\, \underline{r}(y)\, dt\, du\, dx\, dy \tag{42}$$

Using the factoring properties of Gaussian variables, it follows that

$$E\left[L^2 \mid H_0\right] = \left\{E\left[L \mid H_0\right]\right\}^2$$

$$+ 2\, E_0 \int_{T_i}^{T_f} \cdot \cdot \int_{T_i}^{T_f} \underline{r}^T(t)\, \underline{h}_\Delta(t,u)\, \underline{K}_0(u,x)\, \underline{h}_\Delta(x,y)\, \underline{r}(y)\, dt\, du\, dx\, dy \tag{43}$$

$$= \left\{E\left[L \mid H_0\right]\right\}^2 + 2\, \mathrm{Tr} \int_{T_i}^{T_f} \cdot \cdot \int_{T_i}^{T_f} \underline{h}_\Delta(t,u)\, \underline{K}_0(u,x)\, \underline{h}_\Delta(x,y)\, \underline{K}_0(y,t)\, dt\, du\, dx\, dy$$

The variance is just the last term.

Therefore

$$d^2 = \frac{\left[\mathrm{Tr} \int_{T_i}^{T_f} \int_{T_i}^{T_f} \underline{h}_{\Delta}(t,u)\, \underline{K}_s(t,u)\, dt\, du \right]^2}{2\, \mathrm{Tr} \int_{T_i}^{T_f} \cdot \cdot \int_{T_i}^{T_f} \underline{h}_{\Delta}(t,u)\, \underline{K}_0(u,x)\, \underline{h}_{\Delta}(x,y)\, \underline{K}_0(y,t)\, dt\, du\, dx\, dy} \tag{44}$$

The denominator can be simplified by using the expression in Eq. (23) for $\underline{h}_{\Delta}(t,u)$ and $\underline{h}_{\Delta}(x,y)$ and observing that

$$\int_{T_i}^{T_f} \underline{K}_n^{-1}(v,u)\, \underline{K}_0(u,x)\, du = \underline{I}\ \delta(v-x) \tag{45}$$

Performing the integrals, we obtain:

$$d^2 = 2\, \mathrm{Tr} \iint_{T_i}^{T_f} \underline{g}(t,x)\, \underline{g}(x,t)\, dx\, dt \tag{46}$$

C. COMMENTS

In this chapter we derived the structure of the optimum receiver for the case of Gaussian signals in Gaussian noise. To implement it, we must solve Eq. (22) or (24), which follow.

$$\underline{K}_s(t,x) = \iint_{T_i}^{T_f} \left[\underline{K}_s(t,u) + \underline{K}_n(t,u)\right] \underline{h}_{\Delta}(u,z) \left[\underline{K}_n(z,x)\right] du\, dz \qquad T_i \le t,x \le T_f \tag{22}$$

or

$$\underline{K}_s(t,x) = \int_{T_i}^{T_f} \left[\underline{K}_s(t,u) + \underline{K}_n(t,u) \right] \underline{g}(u,x)\, du \qquad T_i < t,\, x < T_f \tag{24}$$

We also found that the output signal-to-noise ratio

$$d^2 = 2\,\mathrm{Tr} \iint_{T_i}^{T_f} \underline{g}(t,x)\, \underline{g}(x,t)\, dx\, dt \tag{46}$$

depended on the solution to Eq. (24). For arbitrary observation intervals and random process statistics the solution is difficult. Explicit solutions can be obtained in several cases of importance:

1. If the signal and noise processes are stationary with rational spectra, one can transform the integral equation into a differential equation, solve it, and substitute the solution into the integral equation to satisfy the boundary conditions. This case is conceptually straightforward but extremely tedious.

2. When the noise is "white," $\underline{K}_n(t,u)$ can be written as:

   $$\underline{K}_n(t,u) = \frac{N_0}{2}\, \underline{I}$$

   If the largest eigenvalue, λ, of $\underline{K}_s(t,u)$ is much less than $N_0/2$, a solution follows easily. This is commonly referred to as the "threshold" or "coherently undetectable" case. One can show that

   $$\underline{h}_{\Delta}(t,u) \simeq \underline{K}_s(t,u).$$

3. If, in addition to the conditions of case (1), we assume that

   $$T = T_f - T_i$$

   is long, then a very simple solution can be obtained using Fourier transform techniques. This case is appropriate in most sonar problems and is developed in detail in the next chapter.

III. LONG OBSERVATION INTERVALS, STATIONARY PROCESSES

A. GENERAL CASE

In this case, simple solutions for the integral equations can be obtained. We assume that the signal and noise processes are wide-sense stationary and that the observation interval is long.

First, we solve the equation for the optimum processor filter $\underline{h}_\Delta(u, z)$. We let $T_i = -T/2$ and $T_f = +T/2$. Then, by Eq. (21),

$$\underline{K}_s(t, x) = \int_{-T/2}^{+T/2}\int \underline{K}_1(t, u)\, \underline{h}_\Delta(u, z)\, \underline{K}_n(z, x)\, du\, dz \tag{47}$$

We then multiply by $\underline{K}_n^{-1}(x, y)$ and integrate with respect to x. In addition, we can write the covariance functions in terms of differences of their arguments:

$$\int_{-T/2}^{+T/2} \underline{K}_s(t - x)\, \underline{K}_n^{-1}(x - y)\, dx = \int_{-T/2}^{+T/2} \underline{K}_1(t - u)\, \underline{h}_\Delta(u - y)\, du \tag{48}$$

Now we define the Fourier transforms of the various matrices. For example,

$$\underline{S}_n(\omega) = \int_{-\infty}^{\infty} d\tau\, e^{-j\omega\tau}\, \underline{K}_n(\tau) \tag{49a}$$

and

$$\underline{K}_n(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{+j\omega\tau}\, \underline{S}_n(\omega) \tag{49b}$$

Observe that

$$\underline{S}_n^T(\omega) = \underline{S}_n^*(\omega) \tag{49c}$$

Next we write $\underline{K}_n^{-1}(\tau)$ and $\underline{h}_L(\tau)$ in terms of their transforms.

$$\int_{-T/2}^{+T/2} dx\, \underline{K}_s(t-x) \int_{-\infty}^{+\infty} \underline{S}_n^{-1}(\omega)\, e^{+j\omega(x-y)}\, \frac{d\omega}{2\pi}$$

$$= \int_{-T/2}^{+T/2} \underline{K}_1(t-x)\, dx \int_{-\infty}^{+\infty} \underline{H}_L(\omega)\, e^{+j\omega(x-y)}\, \frac{d\omega}{2\pi} \tag{50}$$

Re-grouping terms,

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{+j\omega(t-y)} \left(\int_{-T/2}^{+T/2} \underline{K}_s(t-x)\, e^{-j\omega(t-x)} dx \right) \underline{S}_n^{-1}(\omega)$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{+j\omega(t-y)} \left(\int_{-T/2}^{+T/2} \underline{K}_1(t-x)\, e^{-j\omega(t-x)} dx \right) \underline{H}(\omega) \tag{51}$$

As $T \to \infty$, the terms in the parentheses approach $\underline{S}_s(\omega)$ and $\underline{S}_1(\omega)$ respectively. Therefore,

$$\underline{S}_s(\omega)\, \underline{S}_n^{-1}(\omega) = \underline{S}_1(\omega)\, \underline{H}_L(\omega) \tag{52}$$

or

$$\underline{H}_{\Delta}(\omega) = \underline{S}_I^{-1}(\omega)\,\underline{S}_s(\omega)\,\underline{S}_n^{-1}(\omega) \tag{53}$$

From Eq. (23), it follows directly that

$$\underline{G}(\omega) = \underline{S}_I^{-1}(\omega)\,\underline{S}_s(\omega) \tag{54a}$$

and

$$\underline{G}^T(\omega) = \underline{S}_s^*(\omega)\,\underline{S}_I^{-1*}(\omega)$$

This result is well-known from unrealizable Wiener filter theory. The output is:

$$\hat{\underline{\mathcal{S}}}_s(\omega) \stackrel{\Delta}{=} \underline{G}^{T^*}(j\omega)\,\underline{R}(j\omega) \tag{54b†}$$

The test statistic is:

$$L = \int_{-T/2}^{+T/2}\int_{-T/2}^{+T/2} \underline{r}^T(t)\,\underline{h}_{\Delta}(t-u)\,\underline{r}(u)\,dt\,du \tag{55}$$

This can be expressed in the frequency domain as:

$$L = \int_{-\infty}^{\infty} R^{*T}(\omega)\,\underline{H}_{\Delta}(\omega)\,\underline{R}(\omega)\,\frac{d\omega}{2\pi} \tag{56}$$

---

†Here $\underline{R}(\omega)$ is the Fourier transform of $\underline{r}(t)$. To be correct, one should use the integrated transform, but the ordinary transform is more familiar so we use it. A script $\mathcal{S}_s(\omega)$ denotes the Fourier transform of $\underline{s}(t)$, while an ordinary $S_s(\omega)$ denotes the power density spectrum.

Two receiver configurations are shown in Figure 6.

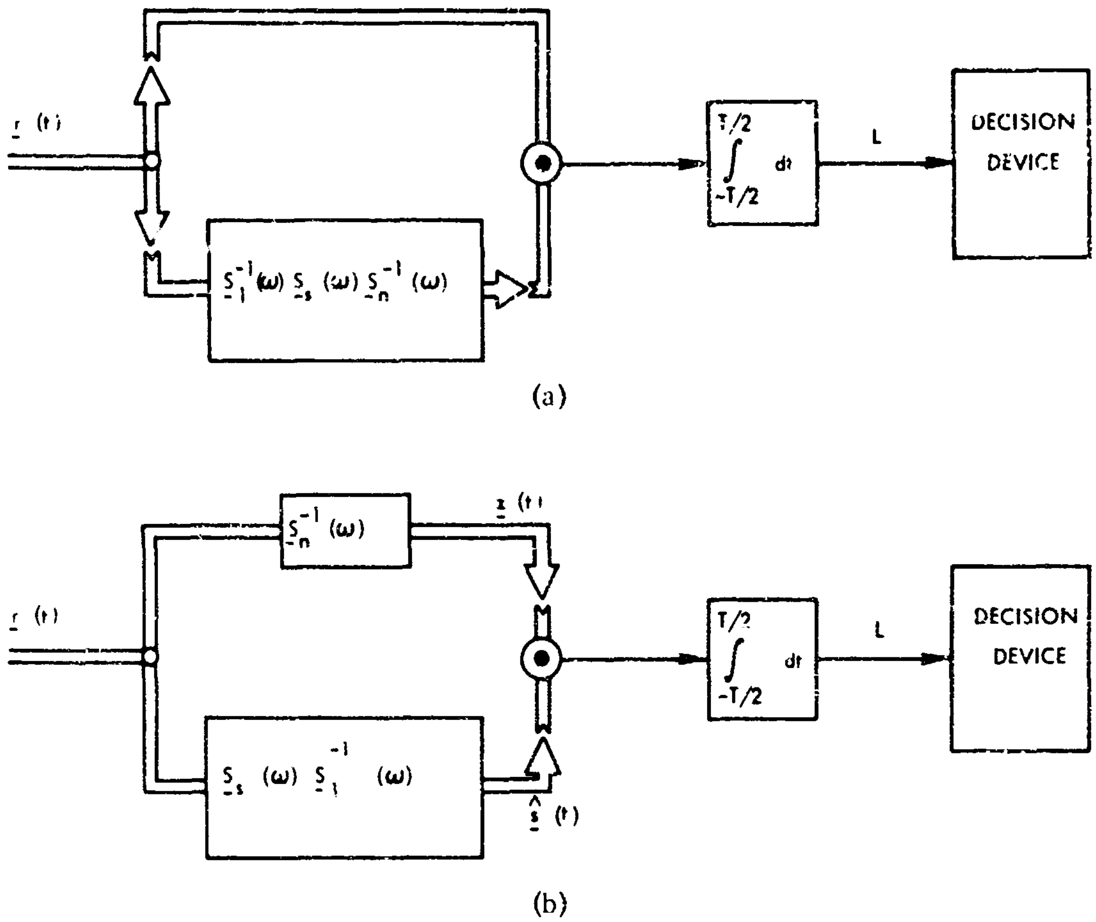

FIGURE 6 OPTIMUM RECEIVER: LONG OBSERVATION INTERVAL

The performance can also be expressed simply. Re-writing Eqs. (44) and (46) in the transform domain, we obtain

$$d^2 = \frac{T}{2} \; \frac{\left\{ \mathrm{Tr} \int_{-\infty}^{\infty} \underline{H}_\Delta(\omega)\, \underline{S}_s(\omega)\, \frac{d\omega}{2\pi} \right\}^2}{\mathrm{Tr} \int_{-\infty}^{\infty} |G(\omega)|^2 \, \frac{d\omega}{2\pi}} \tag{57}$$

In several special cases the results can be simplified.

B. SPECIAL CASES

1. Threshold Case

In the threshold case the input signal-to-noise ratio is low. If

$$S_{s,ij}(\omega) << S_{n,ij}(\omega) \tag{58}$$

for all i, j and ω, then we call the problem a threshold problem. (Eq. (58) is a stronger condition than necessary; some weaker conditions will arise later.) This equation simply says that each element in the signal matrix is smaller than the corresponding element in the noise matrix.

Then,

$$\underline{S}_1(\omega) = \underline{S}_s(\omega) + \underline{S}_n(\omega) \cong \underline{S}_n(\omega) \tag{59}$$

and

$$\underline{H}_\Delta(\omega) \cong \underline{S}_n^{-1}(\omega)\,\underline{S}_s(\omega)\,\underline{S}_n^{-1}(\omega) \tag{60}$$

and

$$\underline{G}(\omega) \cong \underline{S}_n^{-1}(\omega)\,\underline{S}_s(\omega) \tag{61}$$

The performance index d^2 becomes:

$$d^2 = \frac{T}{2} \frac{\left\{ \mathrm{Tr} \int_{-\infty}^{\infty} \underline{S}_n^{-1}(\omega)\,\underline{S}_s(\omega)\,\underline{S}_n^{-1}(\omega)\,\underline{S}_s(\omega)\,\frac{d\omega}{2\pi} \right\}^2}{\left\{ \mathrm{Tr} \int_{-\infty}^{\infty} \underline{S}_n^{-1}(\omega)\,\underline{S}_s(\omega)\,\underline{S}_n^{-1}(\omega)\,\underline{S}_s(\omega)\,\frac{d\omega}{2\pi} \right\}} \tag{62}$$

Cancelling the common term, we have:

$$d^2 = \frac{T}{2} \operatorname{Tr} \int_{-\infty}^{\infty} \underline{S}_n^{-1}(\omega)\, \underline{S}_s(\omega)\, \underline{S}_n^{-1}(\omega)\, \underline{S}_s(\omega) \frac{d\omega}{2\pi} \tag{63}$$

2. Homogeneous Noise, Threshold Case

In this case, the signal spectrum at each hydrophone is identical and the noise spectra are also identical. Then we may write

$$\underline{S}_s(\omega) = S_s(\omega)\, \underline{P}(\omega) \tag{64a}$$

$$\underline{S}_n(\omega) = S_n(\omega)\, \underline{Q}(\omega) \tag{64b}$$

Here $\underline{P}(\omega)$ and $\underline{Q}(\omega)$ represent the cross-spectral terms.

Then,

$$\underline{H}_\Delta(\omega) = \frac{S_s(\omega)}{S_n^2(\omega)}\, \underline{Q}^{-1}(\omega)\, \underline{P}(\omega)\, \underline{Q}^{-1}(\omega) \tag{65}$$

and

$$\underline{G}(\omega) = \frac{S_s(\omega)}{S_n(\omega)}\, \underline{Q}^{-1}(\omega)\, \underline{P}(\omega) \tag{66}$$

and

$$d^2 = \frac{T}{2} \int_{-\infty}^{\infty} \frac{S_s^2(\omega)}{S_n^2(\omega)} \operatorname{Tr} \left[\underline{Q}^{-1}(\omega)\, \underline{P}(\omega)\, \underline{Q}^{-1}(\omega)\, \underline{P}(\omega)\right] \frac{d\omega}{2\pi} \tag{67}$$

3. Single Signal Case

We now turn to the case of principal interest to us. Consider the array shown in Figure 7. The coordinate of each hydrophone is denoted by a vector, $\underline{r}_i$. We assume that the signal component at each hydrophone is identical except for a time delay.

$$\underline{s}(t) = \begin{bmatrix} s(t - \tau_1) \\ s(t - \tau_2) \\ \cdot \\ \cdot \\ \cdot \\ s(t - \tau_N) \end{bmatrix} \tag{68a}$$

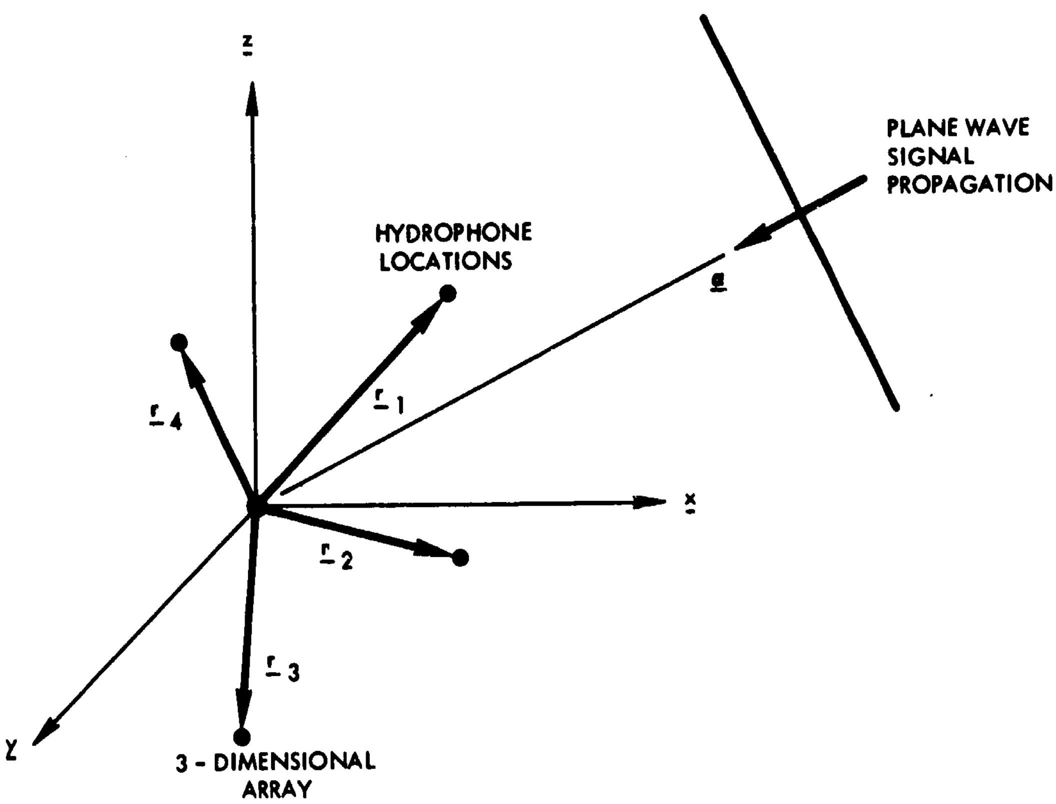

FIGURE 7 HYDROPHONE ARRAY: PLANE WAVE SIGNAL

A simple case in which this assumption is true is the case of a plane wave whose line of propagation is along a vector $\underline{\alpha}$. In this case

$$\tau_i = \frac{\underline{\alpha} \cdot \underline{r}_i}{c} \qquad (68b)$$

We will consider this case in the sequel because of its easy physical interpretation. The modification to account for the general case described by Eq. (68a) is obvious.

Substituting Eq. (68b) into Eq. (68a), we obtain

$$\underline{s}(t) = \begin{bmatrix} s\left(t - \dfrac{\underline{\alpha} \cdot \underline{r}_1}{c}\right) \\ s\left(t - \dfrac{\underline{\alpha} \cdot \underline{r}_2}{c}\right) \\ \cdot \\ \cdot \\ \cdot \\ s\left(t - \dfrac{\underline{\alpha} \cdot \underline{r}_N}{c}\right) \end{bmatrix} \qquad (68c)$$

Then,

$$\underline{S}_s(\omega) = S_s(\omega)\, \underline{P}(\omega) \qquad (69a)$$

where

$$P_{k\ell}(\omega) = e^{+ j \frac{\omega}{c} \underline{\alpha} \cdot [\underline{r}_k - \underline{r}_\ell]} \qquad (69b)$$

In this case, it is easier to perform a preliminary operation on the inputs so that the outputs of the hydrophones are in time synchronization. This corresponds to "steering" the array and is obtained by a set of delay lines, as shown in Figure 8. Observe that we have indicated negative delays; these are obtained physically by including a common positive delay. The noises at the output are a function of the steering direction.

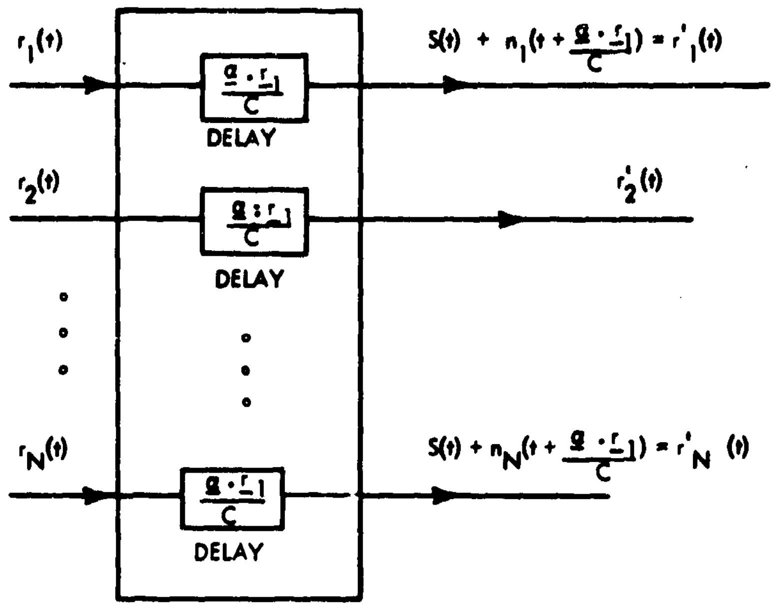

FIGURE 8 ARRAY STEERING

The new noise spectral matrix follows easily.

$$\underline{S}'_{n_{ij}}(\omega) = \underline{S}_{n_{ij}}(\omega)\, e^{+j\frac{\omega}{c}(\underline{\alpha}_j\cdot\underline{r}_j - \underline{\alpha}_i\cdot\underline{r}_i)} \tag{70a}$$

and

$$\underline{S}'_s(\omega) = S_s(\omega)\begin{bmatrix} 1 & 1 & 1 & & 1 \\ \vdots & & \ddots & & \vdots \\ 1 & & & & 1 \end{bmatrix} \tag{70b}$$

From this point on we will regard the steered outputs as our basic input. For notational simplicity, we will leave off the primes.

To realize the filter as shown in Figure 3, we find $\underline{G}(j\omega)$ as given by Eq. (54). Denote the elements in the inverse of $S_1(j\omega)$ as $S_1^{ij}(\omega)$. Then,

$$\underline{G}(j\omega) = \begin{bmatrix} S_1^{11}(\omega) & S_1^{12}(\omega) & \cdots & S_1^{1N}(\omega) \\ \\ S_1^{N1}(\omega) & & & \end{bmatrix} \begin{bmatrix} S_s(\omega) & S_s(\omega) & \cdots & S_s(\omega) \\ \\ S_s(\omega) & S_s(\omega) & & S_s(\omega) \end{bmatrix} \tag{71a}$$

or

$$\underline{G}(j\omega) = S_s(j\omega) \begin{bmatrix} \sum_{j=1}^{N} S_1^{1j}(\omega) & \sum_{j=1}^{N} S_1^{1j}(\omega) & \cdots & \sum_{j=1}^{N} S_1^{1j}(\omega) \\ \sum_{j=1}^{N} S_1^{2j}(\omega) & & & \\ \\ \sum_{j=1}^{N} S_1^{Nj}(\omega) & & & \sum_{j=1}^{N} S_r^{Nj}(\omega) \end{bmatrix} \tag{71b}$$

The output is an N x 1 matrix $\hat{\underline{\lambda}}_s(j\omega)$ which is defined by Eq. (54b).

Substituting Eq. (71b) into Eq. (54b), we find that each element in the matrix is identical. The elements are the estimates of $\lambda_s(\omega)$:

$$\hat{\lambda}_s(\omega) = S_s(j\omega) \sum_{i=1}^{N} \sum_{j=1}^{N} S_1^{ij}(j\omega)\, R_i(j\omega) \tag{72}$$

From Figure 6, we see that we want to form the dot product between $\underline{z}(\omega)$ and $\underline{\hat{s}}_S(j\omega)$. In Figure 9(a) we show the receiver for this particular case using matrix notation. In Figure 9(b), we show the actual operations. Note that each element in $\underline{\hat{s}}_S(j\omega)$ is the same. Therefore, we can add the elements in $\underline{z}(\omega)$ and then multiply the two scalar quantities. The resulting receiver is shown in Figure 10.

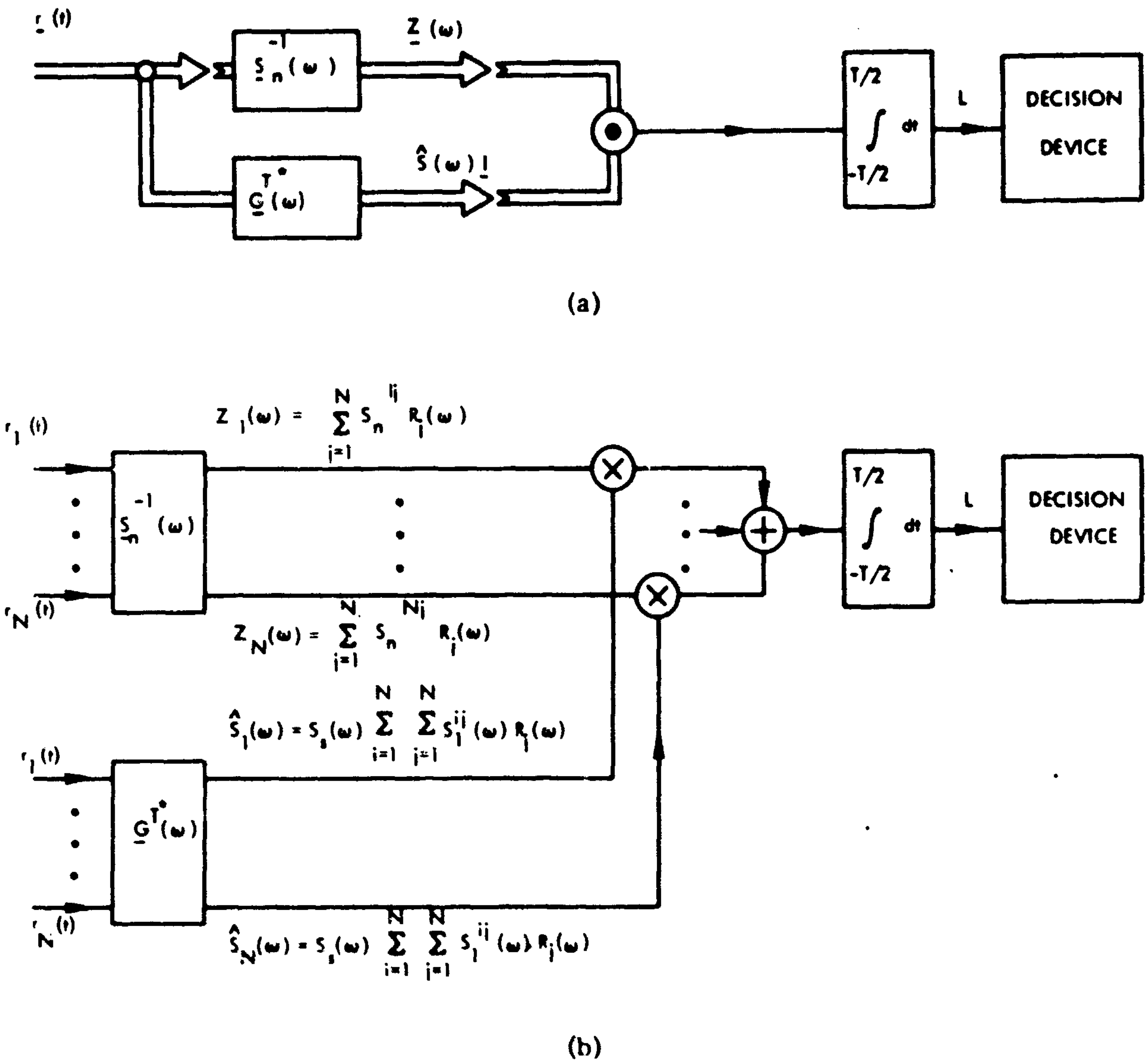

FIGURE 9 OPTIMUM ARRAY PROCESSOR: SINGLE SIGNAL SOURCE

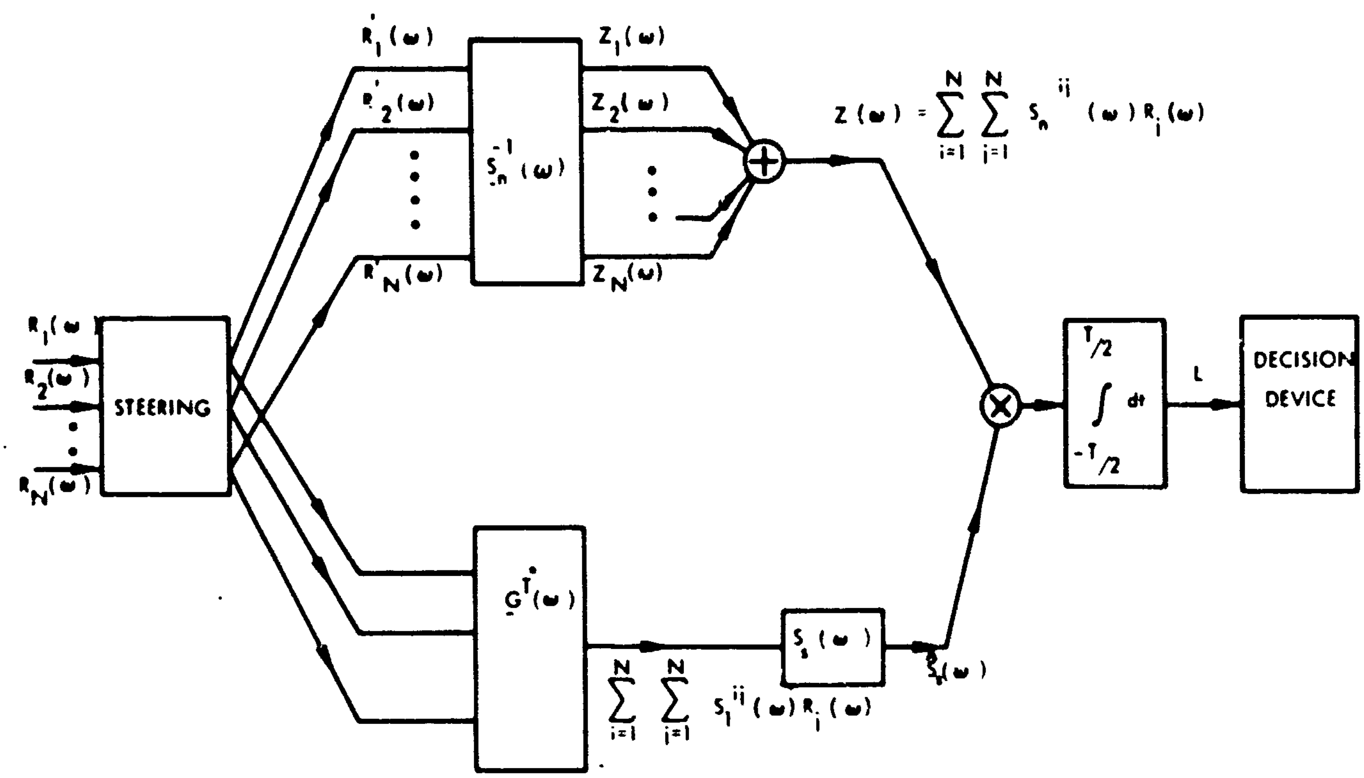

FIGURE 10 OPTIMUM ARRAY PROCESSOR: SINGLE SIGNAL SOURCE (ALTERNATE FORM)

The difficulty with the receiver structure in Figure 10 is that there are two separate combining operations. We will eliminate this difficulty by proving that we can obtain $\hat{\lambda}_s(j\omega)$ by passing $z(j\omega)$ through a (scalar) filter, $F(j\omega)$. In other words, we want to prove:

$$\hat{\lambda}_s(j\omega) = F(j\omega)\, z(j\omega) \tag{73a}$$

or

$$S_s(\omega) \sum_{i=1}^{N} \sum_{j=1}^{N} S_1^{ij}(j\omega)\, R_j(j\omega) = F(j\omega) \sum_{i=1}^{N} \sum_{j=1}^{N} S_n^{ij}(j\omega)\, R_j(j\omega) \tag{73b}$$

In matrix notation,

$$S_s(j\omega)\,\underline{1}^T\,\underline{S}_1^{-1}(j\omega)\,\underline{R}(j\omega) = \underline{1}^T F(j\omega)\,\underline{S}_n^{-1}(j\omega)\,\underline{R}(j\omega) \tag{73c}$$

where

$$\underline{1}^T = [1 \quad 1 \quad . \; . \; . \quad 1] \tag{74}$$

Equivalently, we must prove:

$$S_s(j\omega)\,\underline{1}^T\,\underline{S}_1^{-1}(j\omega) = \underline{1}^T F(j\omega)\,\underline{S}_n^{-1}(j\omega) \tag{75}$$

Post-multiplying by $\underline{S}_1(j\omega)$, we have

$$S_s(j\omega)\,\underline{1}^T = \underline{1}^T F(j\omega)\,\underline{S}_n^{-1}(j\omega)\,\underline{S}_1(j\omega) \tag{76}$$

Using Eq. (70b) and the definition of $\underline{S}_1(\omega)$, we obtain.

$$S_s(j\omega)\,\underline{1}^T = \underline{1}^T F(j\omega)\,S_s(j\omega)\begin{bmatrix} \sum_{j=1}^{N} S_n^{1j} & \sum_{j=1}^{N} S_n^{1j} & \cdots & \sum_{j=1}^{N} S_n^{1j} \\ \sum_{j=1}^{N} S_n^{2j} & & & \\ & & & \\ \sum_{j=1}^{N} S_n^{Nj} & & & \sum_{j=1}^{N} S_n^{Nj} \end{bmatrix} \tag{77}$$

$$+ F(j\omega)\,\underline{1}^T\underline{1}$$

This implies

$$S_s(j\omega) = F(j\omega) S_s(\omega) \sum_{i=1}^{N} \sum_{j=1}^{N} S_n^{ij}(j\omega) + F(j\omega) \tag{78}$$

Defining

$$\Lambda^{-1}(j\omega) \triangleq \sum_{i=1}^{N} \sum_{j=1}^{N} S_n^{ij}(j\omega) \tag{79}$$

we have:

$$F(\omega) = \frac{\Lambda(j\omega) S_s(j\omega)}{S_s(j\omega) + \Lambda(j\omega)} \tag{80}$$

The resulting realization is shown in Figure 11. Alternately, we see that we can realize the receiver in the filter-squarer form as shown in Figures 12a and 12b. (The purpose of the modification in Figure 12b will be apparent in Chapter IV.) In Figure 13, we show the resulting structure under the threshold assumption. (Here, we assume $S_s(\omega) << \Lambda(j\omega)$ for frequencies of interest.)

This completes our discussion of the general receiver structure for the detection problem. In Chapters V, VI, and VII, we will look at some particular problems. Before doing this, we will demonstrate how a similar structure arises when a waveform-estimation approach is used.

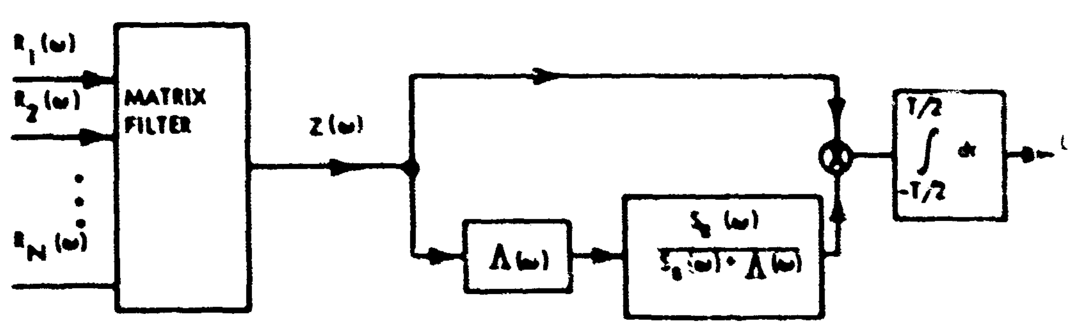

FIGURE 11 SIMPLIFIED FORM OF OPTIMUM ARRAY PROCESSOR

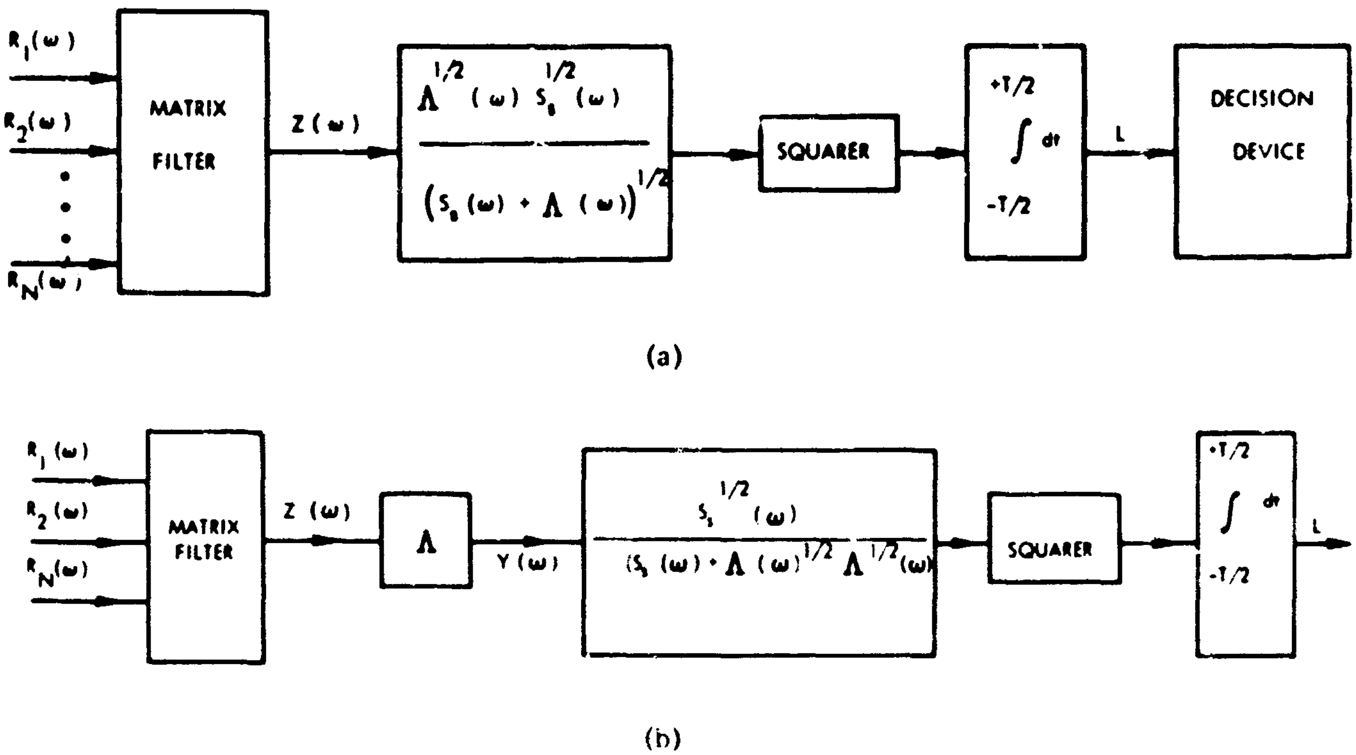

FIGURE 12 FILTER-SQUARER VERSION OF OPTIMUM ARRAY PROCESSOR

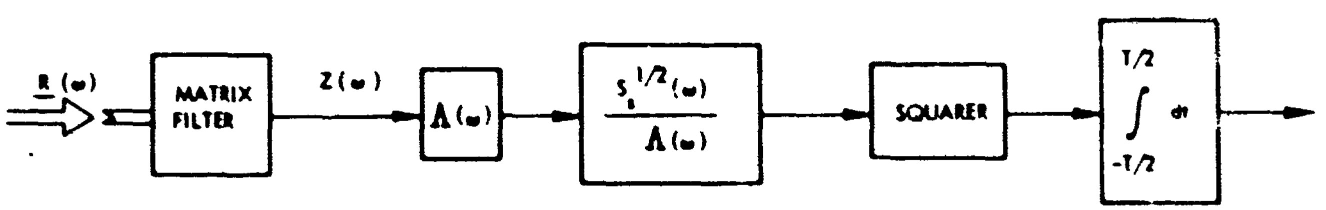

FIGURE 13 THRESHOLD CASE

IV. SIGNAL WAVEFORM ESTIMATION

In many cases detection is not the only question of interest. For example, we may want to process $\underline{r}(t)$ to obtain a good estimate of $s(t)$ to use in a classification problem.

In this chapter, we discuss two possible approaches to waveform estimation. The first is called "distortionless" filtering because the processor is designed so as not to distort the signal. The second is the classical minimum mean-square error approach. After deriving them separately, we show that a matrix MMSE filter can be viewed as a cascade of a matrix distortionless filter and a scalar MMSE filter. This result, which is due to Kelly,(18) is important because it enables us to perform a distortionless combining of the hydrophone outputs and observe it before introducing the signal distortion caused by an MMSE filter.

Finally, we relate the waveform estimation problem to the detection problem and show that the distortionless combining operation is the only matrix operation necessary in both cases.

In this chapter we will assume that the signal is a plane wave whose direction is known. As discussed in Section B-3 of Chapter III, we assume that the delays to steer the array on the target have already been inserted.

Thus, $\underline{r}(t)$ is an N-element column matrix whose elements are:†

$$r_i(t) = s(t) + n_i(t) \tag{81}$$

The spectral matrix is $\underline{S}_n(\omega)$, where:

$$\underline{S}_n(\omega) = \int_{-\infty}^{\infty} E[\underline{n}(t)\,\underline{n}^T(t-\tau)]\, e^{-j\omega\tau}\, d\tau \tag{82}$$

With this model, we first develop the idea of distortionless filtering.

---

†As before, the primes are suppressed for notational simplicity.

A. DISTORTIONLESS FILTERS†

The filter of interest is shown in Figure 14. The outputs of the beam-forming operation are given by Eq. 81. Each output contains the signal s(t) and a noise $n_i(t)$. These inputs are filtered and summed to give a single output y(t).

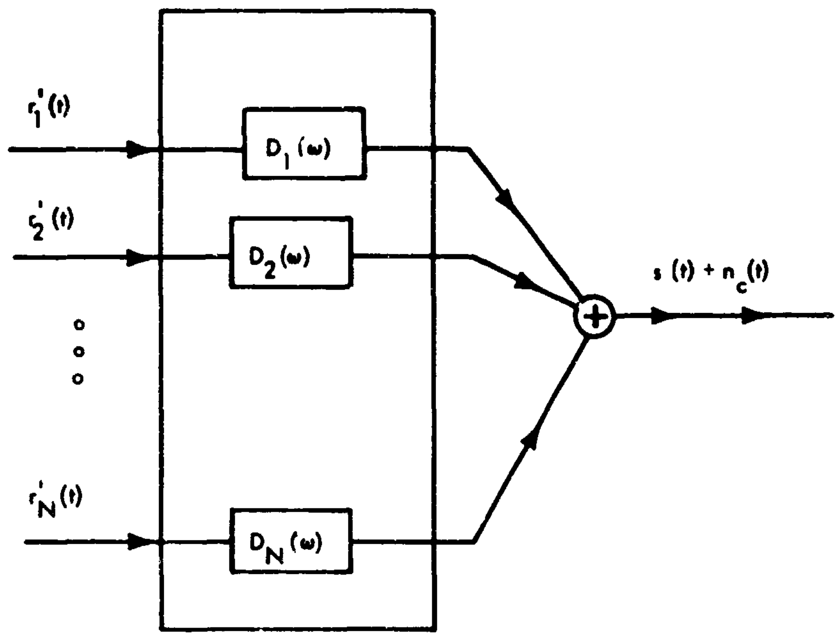

FIGURE 14 DISTORTIONLESS COMBINING

It is required that, in the absence of noise,

$$y(t) = s(t) \tag{83}$$

for <u>any</u> signal s(t).

Under this constraint, we wish to minimize the variance of $n_c(t)$.

---

†The idea of combining multiple inputs in a statistically optimum manner under the constraint of no signal distortion is due to Darlington.(19) An interesting discussion of the method is contained in Brown and Nilsson's text.(20) It was derived independently by Levin(21) as a minimum-variance unbiased estimator. A simple derivation is given by Kelly;(18) our discussion follows this later reference.

The output of the filter is:

$$Y(j\omega) = \underline{D}^T(j\omega)\,\underline{R}(j\omega) = \sum_{i=1}^{N} D_i(j\omega)\,R_i(j\omega) \tag{84}$$

where $\underline{D}(j\omega)$ is N x 1 matrix.

$$\underline{D}(j\omega) = \begin{bmatrix} D_1(j\omega) \\ D_2(j\omega) \\ \vdots \\ D_N(j\omega) \end{bmatrix} \tag{85}$$

The constraint of no distortion implies:

$$\sum_{i=1}^{N} D_i(j\omega) = 1 \tag{86a}$$

or

$$\underline{1}^T\,\underline{D}(j\omega) = \underline{D}^T(j\omega)\,\underline{1} = 1 \tag{86b}$$

for all ω.

The variance is:

$$\sigma^2_{n_c} = E\left[n_c^2(t)\right] = \int_{-\infty}^{\infty} S_{n_c}(\omega)\,\frac{d\omega}{2\pi} \tag{87}$$

Now,

$$S_{n_c}(\omega) = E\left[\underline{D}^T(j\omega)\,\underline{n}_c(j\omega)\,\underline{n}_c^{T^*}(j\omega)\,\underline{D}^*(j\omega)\right] \tag{88a}$$

$$S_{n_c}(\omega) = \underline{D}^T(j\omega)\,\underline{S}_n(j\omega)\,\underline{D}^*(j\omega) \tag{88b}$$

We want to minimize $\sigma^2_{n_c}$ subject to the constraint in Eq. (86b); to do this, we use the Lagrange multiplier technique:

$$F = \int_{-\infty}^{\infty} \left\{\underline{D}^T(j\omega)\,\underline{S}_n(j\omega)\,\underline{D}^*(j\omega) + \lambda\,\underline{1}^T\underline{D}(j\omega)\right\}\frac{d\omega}{2\pi} \tag{89a}$$

Since the integrand is always positive, we can minimize at each frequency.

$$F_\omega \triangleq \underline{D}^T(j\omega)\,\underline{S}_n(j\omega)\,\underline{D}^*(j\omega) + \lambda\,\underline{D}^T(j\omega)\,\underline{1} \tag{89b}$$

Differentiating with respect to $D_r(j\omega)$, the r^{th} element in $D(j\omega)$, we obtain

$$[0\ldots1\ldots]\,\underline{S}_n(j\omega)\,\underline{D}^*(j\omega) + \lambda\,|0\ldots1\ldots]\,\underline{1} = 0 \qquad r = 1, 2, \ldots N \tag{90a}$$

This is equivalent to a single matrix equation.

$$\underline{S}_n(j\omega)\,\underline{D}^*(j\omega) = -\lambda\,1 \tag{90b}$$

Pre-multiplying by $\underline{S}_n^{-1}(j\omega)$, we obtain

$$\underline{D}^*(j\omega) = -\lambda \underline{S}_n^{-1}(j\omega)\, 1 \tag{91a}$$

or

$$\underline{D}(j\omega) = -\lambda \left[\underline{S}_n^{-1}(j\omega)\right]^* \underline{1} \tag{91b}$$

$$= -\lambda \left[\underline{S}_n^{-1}(j\omega)\right]^T \underline{1}$$

or

$$D_j(j\omega) = -\lambda \sum_{i=1}^{N} S_n^{ij}(j\omega) \tag{91c}$$

The value of λ is obtained from the constraint equation (86a):

$$\sum_{j=1}^{N} D_j(j\omega) = -\lambda \sum_{i=1}^{N} \sum_{j=1}^{N} S_n^{ij}(j\omega) = 1 \tag{92}$$

so

$$-\lambda = \left[\sum_{i=1}^{N} \sum_{j=1}^{N} S_n^{ij}(j\omega)\right]^{-1} \triangleq \Lambda(j\omega) \tag{93}$$

Therefore,

$$D_j(j\omega) = \Lambda(j\omega) \sum_{i=1}^{N} S_n^{ij}(j\omega) \tag{94a}$$

The distortionless combiner is shown in Figure 15.

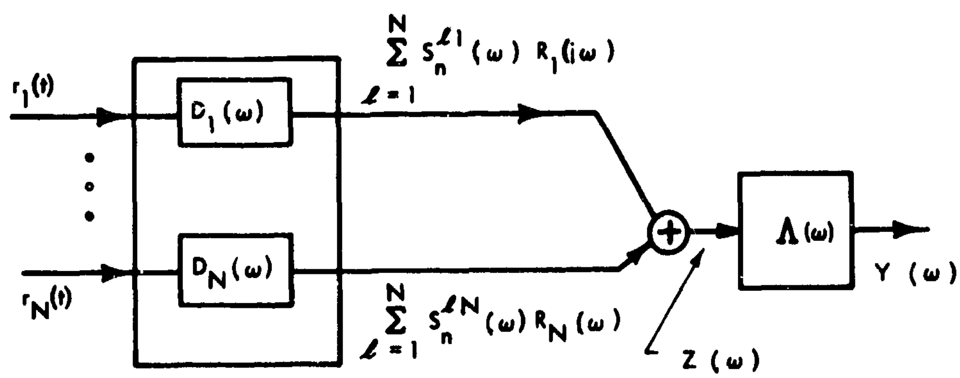

FIGURE 15 OPTIMUM COMBINER

We see that the signal $z(j\omega)$ is identical to the combiner output in Figure 11. Similarly, we see that the distortionless waveform $Y(j\omega)$ also appears in Figures 12 and 13b.

$$Y(j\omega) = \frac{\sum_{\ell=1}^{N} \sum_{j=1}^{N} S_n^{\ell j}(\omega) R_j(j\omega)}{\sum_{i=1}^{N} \sum_{j=1}^{N} S_n^{ij}(\omega)} \tag{94b}$$

The variance using distortionless processing follows easily. Substituting Eq. (94) into Eq. (91b) and the result into Eq. (88b), we have:

$$S_{n_c}(\omega) = \Lambda(\omega)\, \underline{1}^T \underline{S}_n^{-1}(j\omega) \underline{S}_n(j\omega) \underline{S}_n^{-1}(j\omega)\, \underline{1}\, \Lambda(\omega) \tag{95}$$

But

$$\underline{1}^T \underline{S}_n^{-1}(j\omega)\, \underline{1} = (\Lambda(j\omega))^{-1} \tag{96}$$

Therefore,

$$S_{n_c}(\omega) = \Lambda(j\omega) \tag{97}$$

and

$$\int_{-\infty}^{\infty} \Lambda(\omega) \frac{d\omega}{2\pi} = \sigma_c^2 \tag{98}$$

Several observations now follow easily:

(1) Looking at Figure 11, we see that $\hat{s}(t)$ is obtained by operating on the distortionless output $y(t)$ by a filter

$$\frac{S_s(\omega)}{S_s(\omega) + \Lambda(\omega)} \tag{99a}$$

This is precisely the optimum unrealizable Wiener filter for estimating $s(t)$ using a minimum mean-square error criterion when the input is $y(t)$. (Since the "noise" spectrum is $\Lambda(\omega)$). The mean-square error is

$$\xi_{mmse} = \int_{-\infty}^{\infty} \frac{\Lambda(\omega)\, S_s(\omega)}{S_s(\omega) + \Lambda(\omega)} \frac{d\omega}{2\pi} \tag{99b}$$

(2) All of the processing up to $y(t)$ in Figures 12, 13b, and 16 is independent of the signal. The effect is a noise-reduction due to combining. In Figure 16 we re-draw the receivers in Figure 12 and 13b to emphasize the distortionless signal.

Before leaving waveform estimation, we will consider the problem from a different viewpoint.

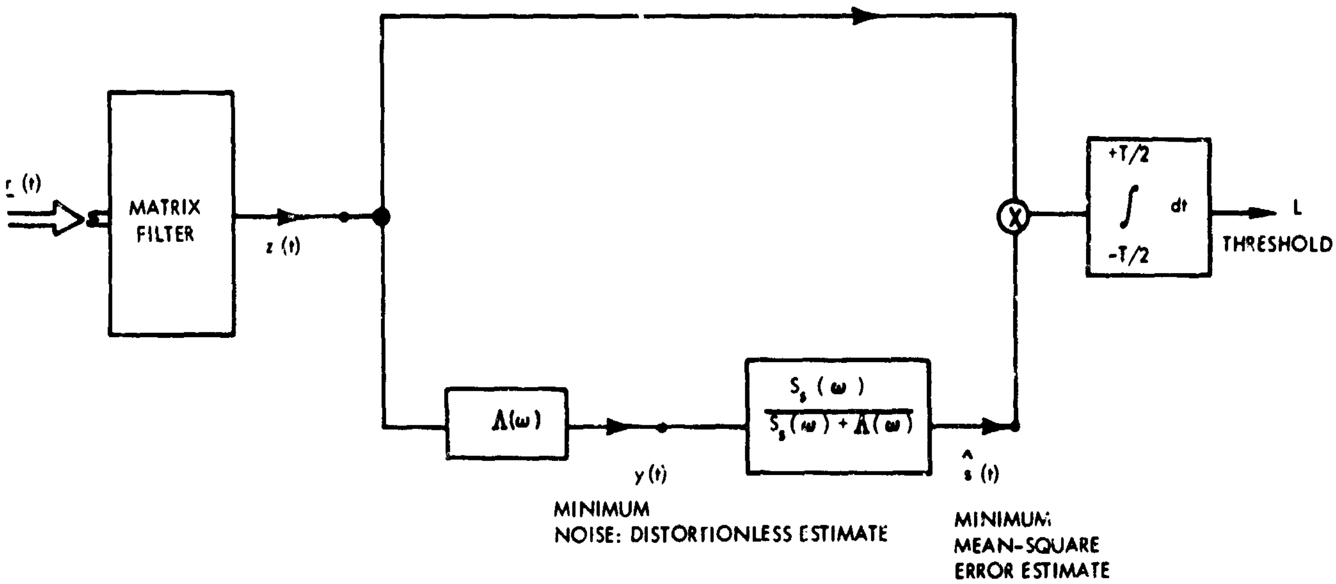

FIGURE 16a ESTIMATOR-CORRELATOR RECEIVER

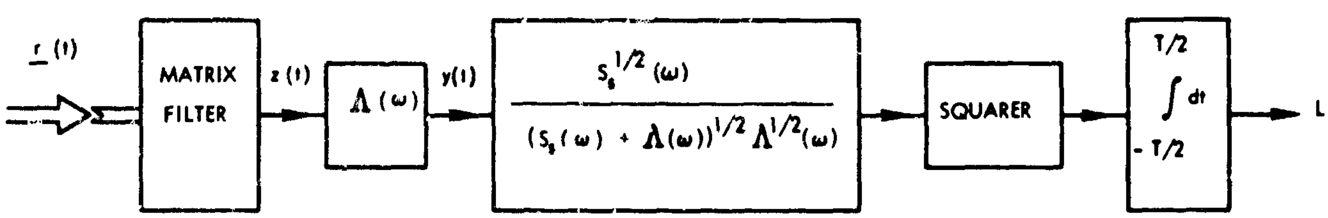

FIGURE 16b FILTER-SQUARER RECEIVER

B. MAXIMUM LIKELIHOOD ESTIMATES AND GENERALIZED LIKELIHOOD RATIO TESTS

The derivation of the optimum distortionless filter started with a rather arbitrary linear filter restriction. One can also approach the problem from the standpoint of a maximum likelihood estimate of $s(t)$.†

The likelihood function of $\underline{r}(t)$, given $s(t)$, is

$$
\begin{aligned}
\ln \Lambda(\underline{r}(t):s(t)) = &+ \frac{1}{2}\int_{-\infty}^{\infty} s^*(\omega)\,\underline{1}^T \underline{S}_n^{-1}(\omega)\,\underline{R}(\omega)\,\frac{d\omega}{2\pi} \\
&+ \frac{1}{2}\int_{-\infty}^{\infty} \underline{R}^{*T}(\omega)\,\underline{S}_n^{-1}(\omega)\,\underline{1}\,s(\omega)\,\frac{d\omega}{2\pi} \qquad (100a) \\
&- \int_{-\infty}^{\infty} s^*(\omega)\,\underline{1}^T \underline{S}_n^{-1}(\omega)\,\underline{1}\,s(\omega)\,\frac{d\omega}{2\pi}
\end{aligned}
$$

where we have assumed an infinite observation interval and stationary processes.

The maximum likelihood estimate of $s(\omega)$ is simply the function $\hat{s}_{m\ell}(\omega)$ that maximizes the likelihood function. To find it, we set

$$
s(\omega) = \hat{s}_{m\ell}(\omega) + \epsilon\, s_\epsilon(\omega) \quad , \qquad (100b)
$$

differentiate $\ln\Lambda$ with respect to ϵ, and require the result to equal zero for $\epsilon = 0$ and all $s_\epsilon(\omega)$. This gives

$$
\begin{aligned}
0 = &\int_{-\infty}^{\infty} s_\epsilon^*(\omega)\,\frac{d\omega}{2\pi}\left\{\underline{1}^T \underline{S}_n^{-1}(\omega)\,\underline{R}(\omega) - \underline{1}^T \underline{S}_n^{-1}(\omega)\,\underline{1}\,\hat{s}_{m\ell}(\omega)\right\} \\
&+ \int_{-\infty}^{\infty} s_\epsilon(\omega)\,\frac{d\omega}{2\pi}\left\{\underline{R}^{*T}(\omega)\,\underline{S}_n^{-1}(\omega)\,\underline{1} - \hat{s}^*\,\underline{1}^T \underline{S}_n^{-1}(\omega)\,\underline{1}\right\} \qquad (100c)
\end{aligned}
$$

†The results through Eq. 100c were previously derived by Kelly and Levin.(22)

The two bracketed terms are conjugates. Since $\lambda_\epsilon(\omega)$ is arbitrary, the term inside the bracket must equal zero for all ω. Therefore,

$$\hat{\lambda}_{m\ell}(\omega)\, \underline{1}^T \underline{S}_n^{-1}(\omega)\, \underline{1} = \underline{1}^T \underline{S}_n^{-1}(\omega)\, \underline{R}(\omega) \tag{100d}$$

Writing Eq. (100d) out, we have:

$$\hat{\lambda}_{m\ell}(j\omega) = \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} S_n^{ij}(\omega)\, R_j(\omega)}{\sum_{i=1}^{N} \sum_{j=1}^{N} S_n^{ij}(\omega)} \tag{100e}$$

Looking at Eq. (94b), we see that $\hat{\lambda}_{m\ell}(j\omega)$ is identical to $Y(j\omega)$.

Observe that there was no a priori restriction to a linear processor.

The detection analog to the maximum likelihood estimate is the generalized likelihood ratio test (e.g., Davenport and Root, Chap. 14[1] or Van Trees, Section 2.5.[5])

The generalized likelihood ratio test is

$$\Lambda_g(\underline{r}(t)) = \frac{\max\limits_{s(t)} p\left[\underline{r}(t) \mid H_1\right]}{p\left[\underline{r}(t) \mid H_0\right]} \gtrless \nu \tag{101a}$$

The numerator is simply $p\left[\underline{r}(t) \mid H_1\right]$ evaluated at $s(t) = \hat{s}_{m\ell}(t)$

Substituting Eq. (100e) into Eq. (100a) gives $\ell n\, \Lambda_g(\underline{r}(t))$.

$$\ln \Lambda_g(\underline{r}(t)) = +\frac{1}{2}\int_{-\infty}^{\infty} \hat{\lambda}^*_{m\ell}(\omega)\, \underline{1}^T \underline{S}_n^{-1}(\omega)\, \underline{R}(\omega)\frac{d\omega}{2\pi}$$

$$+\frac{1}{2}\int_{-\infty}^{\infty} \underline{R}^{*T}(\omega)\, \underline{S}_n^{-1}(\omega)\, \underline{1}\, \hat{\lambda}_{m\ell}(\omega)\frac{d\omega}{2\pi} \gtrless \gamma' \qquad (101b)$$

(We incorporated the last term into the threshold.)

Now,

$$\left.\begin{aligned} \hat{\lambda}_{m\ell}(\omega) &= \Lambda(\omega)\, \underline{1}^T \underline{S}_n^{-1}(\omega)\, \underline{R}(\omega) \\ &= \Lambda(\omega)\, \underline{R}^T(\omega) \underline{S}_n^{-1\,T}(\omega)\, \underline{1} \\ &= \Lambda(\omega) \underline{R}^T(\omega) \underline{S}_n^{-1*}(\omega)\, \underline{1} \end{aligned}\right\} \qquad (101c)$$

Substituting Eq. (101c) into Eq. (101b), we obtain

$$\ln \Lambda_g(\underline{r}(t)) = +\frac{1}{2}\int_{-\infty}^{\infty} \hat{\lambda}^*_{m\ell}(\omega)\, \frac{\hat{\lambda}_{m\ell}(\omega)}{\Lambda(\omega)}\, \frac{d\omega}{2\pi}$$

$$+\frac{1}{2}\int_{-\infty}^{\infty} \frac{\hat{\lambda}^*_{m\ell}(\omega)}{\Lambda(\omega)}\, \hat{\lambda}_{m\ell}(\omega)\, \frac{d\omega}{2\pi} \qquad (101d)$$

(Recall $\Lambda(\omega)$ is real)

Therefore,

$$\ell n \Lambda_g(\underline{r}(t)) = \int_{-\infty}^{\infty} \left| \frac{\hat{m}_\ell(\omega)}{\Lambda^{1/2}(\omega)} \right| \frac{d\omega}{2\pi} \qquad (101e)$$

Using Parseval's theorem, we obtain the receiver for the generalized likelihood ratio test that is shown in Figure 17.

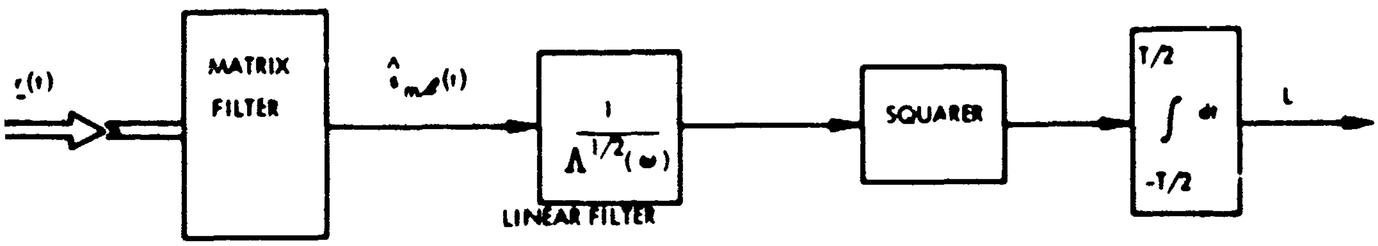

FIGURE 17 RECEIVER TO IMPLEMENT GENERALIZED LIKELIHOOD RATIO TEST

C. SUMMARY

In this chapter, we have related the ideas of distortionless filtering and maximum likelihood estimation. Criteria of this type are appropriate when the signal is a non-random, but unknown, waveform. Further, we found that the generalized likelihood ratio test is closely related to this estimation procedure.

Combining these results with those of Chapter III, we obtain the composite receiver shown in Figure 18.

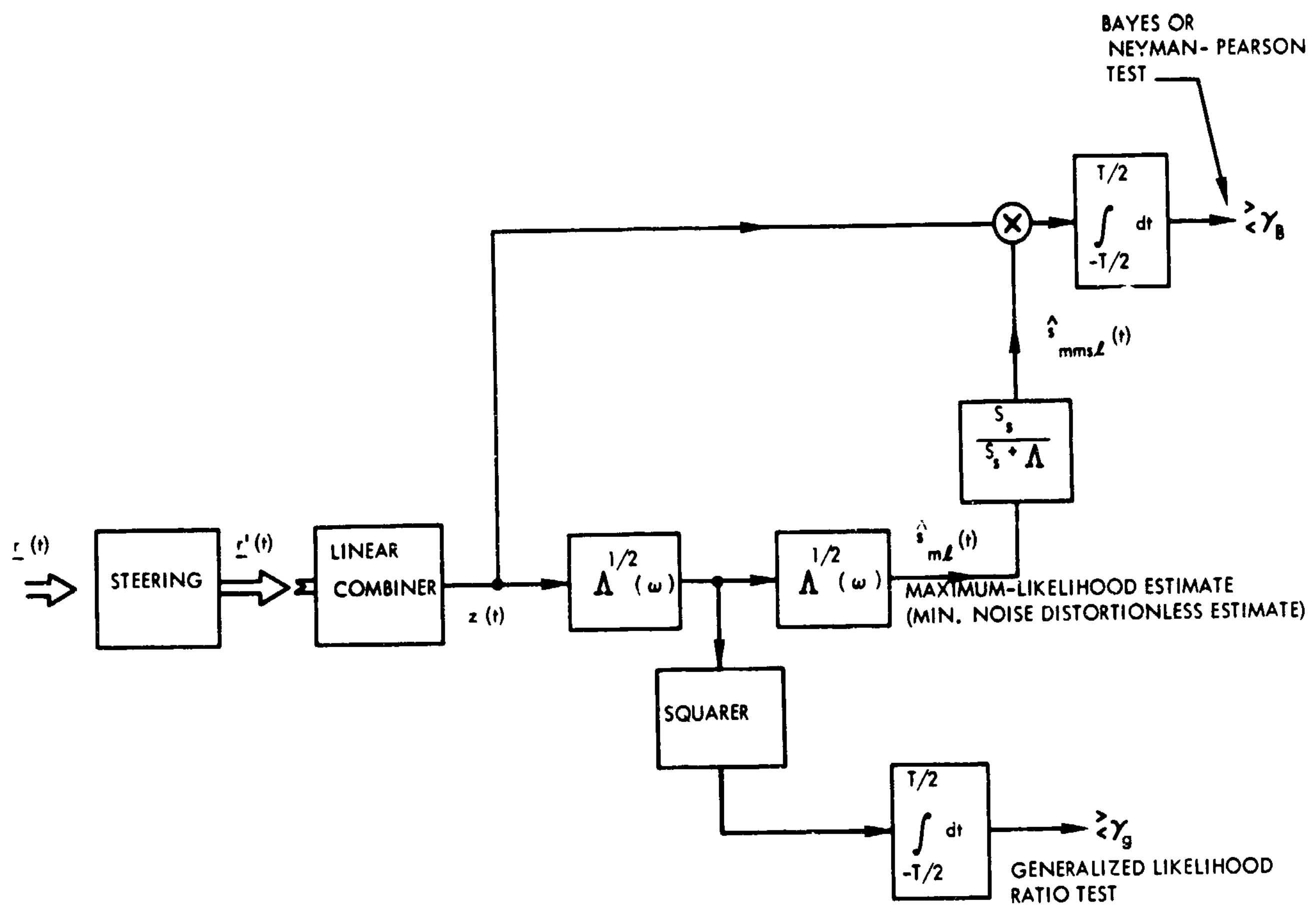

FIGURE 18 CRITERION INVARIANT RECEIVER

V. HOMOGENEOUS NOISE: ARRAY GAIN

A special case of interest is when the noise is homogeneous. Specifically,

$$S_{n_{11}}(j\omega) = S_{n_{ii}}(j\omega) \triangleq S_n(j\omega) \tag{102a}$$

and

$$S_{n_{ij}}(j\omega) = S_n(j\omega)\,\rho_{ij}(j\omega) \tag{102b}$$

Let

$\underline{\rho}(j\omega)$ be a normalized cross-spectral matrix with elements $\rho_{ij}(j\omega)$

Then,

$$\underline{S}_n^{-1}(j\omega) = (S_n(j\omega))^{-1}\,\underline{\rho}^{-1}(j\omega) \tag{102c}$$

One realization of the resulting receiver is shown in Figure 19.

In this case, it is easy to evaluate the effect of the array on the performance of the system with respect to the following three criteria:

(1) Detection performance index (Eq. (67))

(2) Distortionless signal; minimum noise variance (Eq. (98))

(3) Minimum-mean-square filtering error (Eq. (99b))

We shall consider these three cases in order. At a single hydrophone:

$$d^2 = \frac{T}{2}\int_{-\infty}^{\infty}\left[\frac{S_s(\omega)}{S_n(\omega)}\right]^2 \frac{d\omega}{2\pi} \tag{103}$$

This is simply the scalar version of Eq. (67).

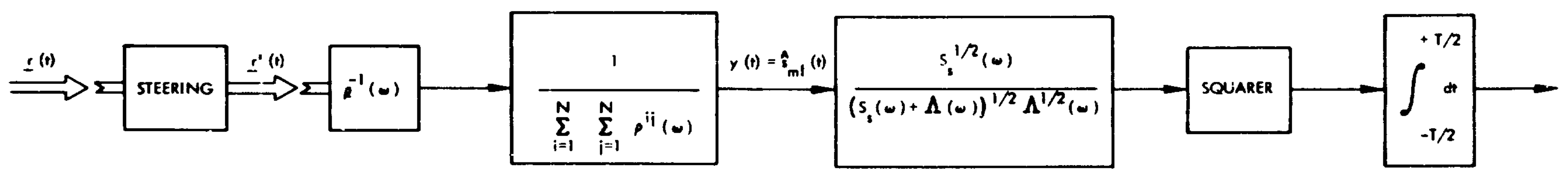

FIGURE 19 FILTER-SQUARER RECEIVER, HOMOGENEOUS NOISE

To evaluate d^2, we look at the signal spectrum of y(t) in Figure 19.† The signal spectrum is $S_s(\omega)$, and the noise spectrum is $\Lambda(\omega)$. Therefore,

$$d^2 = \frac{T}{2} \int_{-\infty}^{\infty} \frac{S_s^2(\omega)}{\Lambda^2(\omega)} \frac{d\omega}{2\pi} \tag{104}$$

but

$$\Lambda^{-1}(\omega) = \sum_{i=1}^{N} \sum_{j=1}^{N} \rho^{ij}(j\omega) \frac{1}{S_n(\omega)} \triangleq \frac{A(\omega)}{S_n(\omega)} \tag{105}$$

Therefore,

$$d^2 = \frac{T}{2} \int_{-\infty}^{\infty} \frac{S_s^2(\omega)}{S_n^2(\omega)} A^2(\omega) \frac{d\omega}{2\pi} \tag{106}$$

We see that the effect of the array is contained completely in the function $A(\omega)$. This function is commonly referred to as the <u>array gain</u>

$$A(\omega) \triangleq \frac{S_n(\omega)}{\Lambda(\omega)} = \sum_{i=1}^{N} \sum_{j=1}^{N} \rho^{ij}(\omega) \tag{107}$$

For independent noises, the array gain is simply N.

We recall that the elements $\rho^{ij}(\omega)$ incorporate the effects of array steering. In terms of the unsteered noise matrix,

$$A(\omega) = \sum_{i}^{N} \sum_{j}^{N} e^{-j \frac{\omega}{c} \underline{\alpha} \cdot [\underline{r}_i - \underline{r}_j]} \rho_o^{ij}(\omega) \tag{108}$$

†We could also work directly with Eq. (67) (see Appendix C).

Thus, unless the noise matrix is diagonal, the array gain is a function of the steering angle.

The array gain also arises naturally in the other cases.

For distortionless filtering, the variance at a single hydrophone is:

$$\sigma_n^2 = \int_{-\infty}^{\infty} S_n(\omega) \frac{d\omega}{2\pi} \tag{109}$$

At the combiner output,

$$\sigma_{n_c}^2 = \int_{-\infty}^{\infty} \Lambda(\omega) \frac{d\omega}{2\pi} = \int_{-\infty}^{\infty} \frac{S_n(\omega)}{\Lambda(\omega)} \frac{d\omega}{2\pi} \tag{110}$$

For minimum-mean-square error filtering, the error at a single hydrophone is:

$$\sigma_T^2 = \int_{-\infty}^{\infty} \frac{S_s(\omega)\, S_n(\omega)}{S_s(\omega) + S_n(\omega)} \frac{d\omega}{2\pi} \tag{111}$$

At the final output,

$$\begin{aligned} \sigma_{T_c}^2 &= \int_{-\infty}^{\infty} \frac{S_s(\omega)\, \Lambda(\omega)}{S_s(\omega) + \Lambda(\omega)} \frac{d\omega}{2\pi} \\ &= \int_{-\infty}^{\infty} \frac{S_s(\omega) \left(\frac{S_n(\omega)}{A(\omega)} \right)}{S_s(\omega) + \frac{S_n(\omega)}{A(\omega)}} \frac{d\omega}{2\pi} \end{aligned} \tag{112}$$

Therefore, for any of the three purposes, the array gain completely characterizes the effect of the array. It is dependent on frequency and steering angle, and its effect is to reduce the noise.

In the next section we derive various noise models and find the array gain for some interesting configurations.

VI. DISTRIBUTED NOISE FIELDS

In Chapters VI and VII we apply the preceding results to simple examples. In this chapter, we shall consider noise fields that are distributed over a region in space.*

A. HYDROPHONE NOISE ONLY

Example 1

In this case, the only noise is hydrophone noise, which is assumed to be independent from one hydrophone to the next. For simplicity, we first assume that the noise spectra are identical. Thus,

$$S_{n,ij}(\omega) = S_n(\omega)\ \delta_{ij} \tag{113}$$

In this particular case, it is clear that the spectra after steering are identical.

Thus,

$$S'_{n,ij}(\omega) = S_{n,ij}(\omega) \tag{114}$$

Clearly,

$$\underline{S}_n^{-1}(\omega) = \frac{1}{S_n(\omega)}\ \underline{I} \tag{115}$$

Combining consists of a simple summing operation.

The array gain equals the number of elements in the array:

$$A_o(\omega) = N \tag{116}$$

Since all of the noise is assumed to be hydrophone noise, the spacing and location of the elements are unimportant.

The modification to include different noise levels is straightforward.

*As pointed out in the references,the results in this chapter are due to Dr. E.J. Kelly, Jr. His contribution is important to the over-all unity of the report, and we are happy to acknowledge it.

B. ISOTROPIC NOISE ONLY

Several models of noise geometry lead to what is called an isotropic noise field. Two of these are:

(1) The noise is assumed to consist of uncorrelated plane waves with identical statistics coming from all directions (e.g., Marsh[23]).

(2) The noise is generated by uncorrelated noise sources uniformly distributed on the surface of a large sphere (e.g., Faran and Hills[24]).

If the noise is assumed to consist of a single frequency, then it is easy to show that

$$\rho_{ij}(\tau) = \frac{\sin kd_{ij}}{kd_{ij}} \cos 2\pi f_0 \tau \qquad (117)$$

where

$k = \dfrac{2\pi f_0}{c}$ is the wave number

c is the velocity of sound

f_0 is the frequency

and

d_{ij} is the distance between elements

Due to the spherical distribution assumption, the correlation function (before steering) does not depend on the orientation between the two elements.

A simple extension of the single-frequency case is when the noise spectrum at each point on the sphere is the same (say $S_n(\omega)$). Then, it follows easily that

$$\rho_{ij}(j\omega) = \frac{\sin \omega \frac{d_{ij}}{c}}{\omega \frac{d_{ij}}{c}} S_n(\omega) \qquad (118)$$

The correlation function may be found, but it is not necessary.

Example 2

To illustrate the isotropic noise case, we consider a simple example due to Bryn[13]. The array is shown in Figure 20.

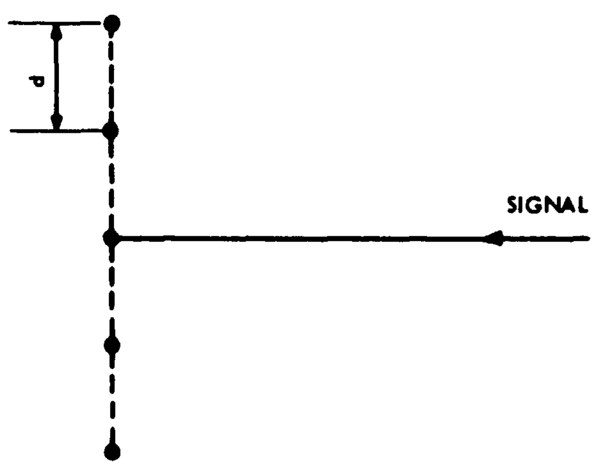

FIGURE 20 UNIFORMLY SPACED FIVE-ELEMENT ARRAY

It consists of five elements arranged in a line. The signal direction is perpendicular to the array. In this case, no delays are necessary for steering. The normalized spectral matrix is:

$$\underline{\rho}(j\omega) = \begin{bmatrix} 1 & \frac{\sin u}{u} & \frac{\sin 2u}{2u} & \frac{\sin 3u}{3u} & \frac{\sin 4u}{4u} \\ & 1 & \frac{\sin u}{u} & \frac{\sin 2u}{2u} & \frac{\sin 3u}{3u} \\ & & 1 & \frac{\sin u}{u} & \frac{\sin 2u}{2u} \\ \text{(symmetrical)} & & & 1 & \frac{\sin u}{u} \\ & & & & 1 \end{bmatrix} \tag{119a}$$

where

$$u = \frac{2\pi f d}{c} = 2\pi \frac{d}{\lambda} \tag{119b}$$

The array gain, $A_o(f)$, as a function of u is shown in Figure 21. The gain obtained by a straight sum is also shown in the figure. We see that for spacings greater than half a wavelength the two are essentially identical.

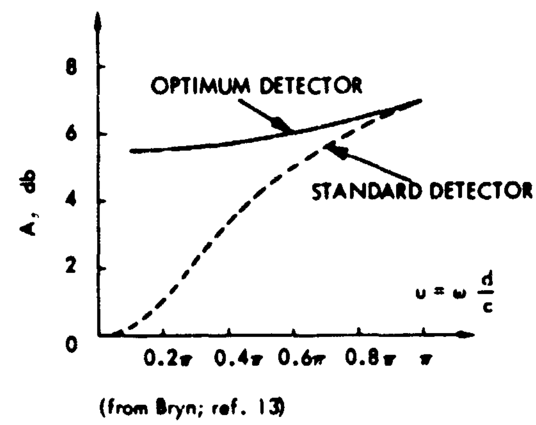

FIGURE 21 ARRAY GAINS

In this model the gains correspond to those obtained by Pritchard[25] using a signal-to-noise ratio criterion. It turns out that the gains are large and opposite. This type of array is commonly referred to as a "super-gain" array; as one would expect, such arrays are sensitive to variations in the gains and therefore are difficult to realize in practice. If one considers more general arrays (the elements not necessarily in a line), then one can get a singular detection problem for the isotropic noise model (see Gaarder[26]). One can eliminate this sensitivity and the possibility of singularity by keeping the hydrophone noise at a non-zero level.

Other array gain calculations have been made (e.g., B. Cron and C. Becker[27]).

We now turn to a second category of noise.

VII. DIRECTIONAL NOISE SOURCES

Frequently, the noise that we want to combat has a strong directional character. The limiting form of this category is a noise plane wave coming from a single direction. We investigate this limiting case under the assumption that the direction is known exactly. If the direction were not known, the receiver would have to measure it. The results for the known direction provide a bound on how well a receiver incorporating measurement could do.

A simple model of directional interference is shown in Figure 22.

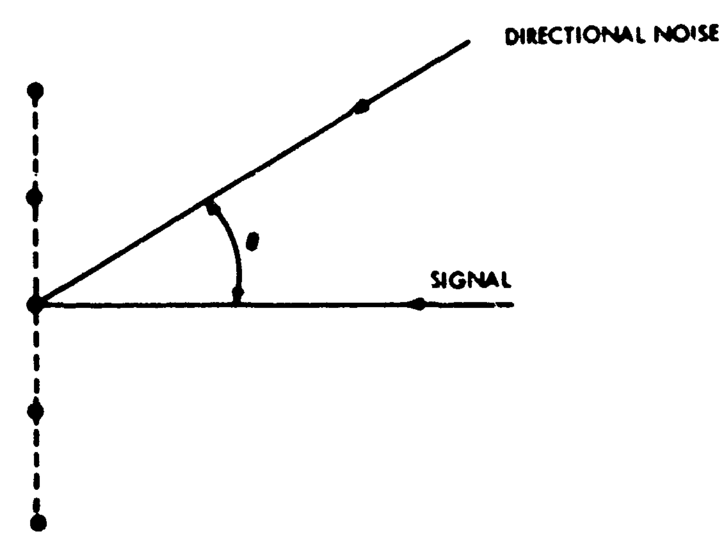

FIGURE 22 DIRECTIONAL NOISE MODEL

The noise is a stationary random process from a known angle θ. For simplicity, we assume that the signal direction is broadside. The array elements are uniformly spaced in a line. If we label the noise voltage at the center element $n(t)$, then for an array with $2N+1$ elements,

$$\underline{n}(t) = \begin{bmatrix} n(t+N\tau) \\ \cdot \\ \cdot \\ \cdot \\ n(t+2\tau) \\ n(t+\tau) \\ n(t) \\ n(t-\tau) \\ \cdot \\ \cdot \\ \cdot \\ n(t-N\tau) \end{bmatrix} \tag{120}$$

where

$$\tau = d/c \sin\theta$$

The noise cross-spectral matrix is easy to find. For a three-element array,

$$\underline{S}_{n_d}(\omega) = S_d(\omega) \begin{bmatrix} 1 & e^{-j\omega\tau} & e^{-j\omega 2\tau} \\ e^{+j\omega\tau} & 1 & e^{-j\omega\tau} \\ e^{+j\omega 2\tau} & e^{+j\omega\tau} & 1 \end{bmatrix} \tag{121}$$

It is clear that this matrix is singular. For any value of θ except zero we can achieve perfect detection. Under H_1, for example,

$$r_1(t-\tau) - r_2(t) = s(t-\tau) - s(t) \tag{122a}$$

and under H_0,

$$r_1(t-\tau) - r_2(t) = 0 \tag{122b}$$

which gives a perfect detection capability.

Therefore, to make the pure directional noise problem meaningful we must include a white noise component. Then,

$$\underline{S}_n(\omega) = \frac{N_0}{2}\underline{I} + \underline{S}_{n_d}(\omega) \tag{123}$$

For the three-element case,

$$\underline{S}(\omega) = \left(\frac{N_0}{2} + S_d(\omega)\right) \begin{bmatrix} 1 & xe^{-j\omega\tau} & xe^{-j\omega 2\tau} \\ xe^{+j\omega\tau} & 1 & xe^{-j\omega\tau} \\ xe^{+j2\omega\tau} & xe^{+j\omega\tau} & 1 \end{bmatrix} \tag{124}$$

where

$$x \triangleq \frac{S_d(\omega)}{\frac{N_0}{2} + S_d(\omega)} \tag{125}$$

The array gain is the sum of all the elements in the inverse of the matrix in Eq. (124). Taking the inverse and summing the elements, we obtain:

$$A(j\omega) = \frac{3(1-x^2) + 2x(x-1)\ [2\cos\omega\tau + \cos 2\omega\tau]}{1 + 2x^3 - 3x^2} \tag{126}$$

As we would expect, for $x = 0$

$$A(j\omega) = 3$$

To indicate the behavior, we have plotted $A(j\omega)$ for two values of $\omega\tau$ in Figure 23.

For any given element spacing, these can be translated to a particular value of θ.

For example, if $d = \lambda/2$, then

$$\omega\tau = \pi \sin\theta \tag{127}$$

and the two curves shown represent $\theta = 0$ and $30°$, respectively. We have also indicated the array gain, $A_c(j\omega)$ for a conventional array which sums the outputs.

The procedure for finding the array gain for larger arrays or multiple directional noise sources is conceptually straightforward, and the actual computation can be done numerically.

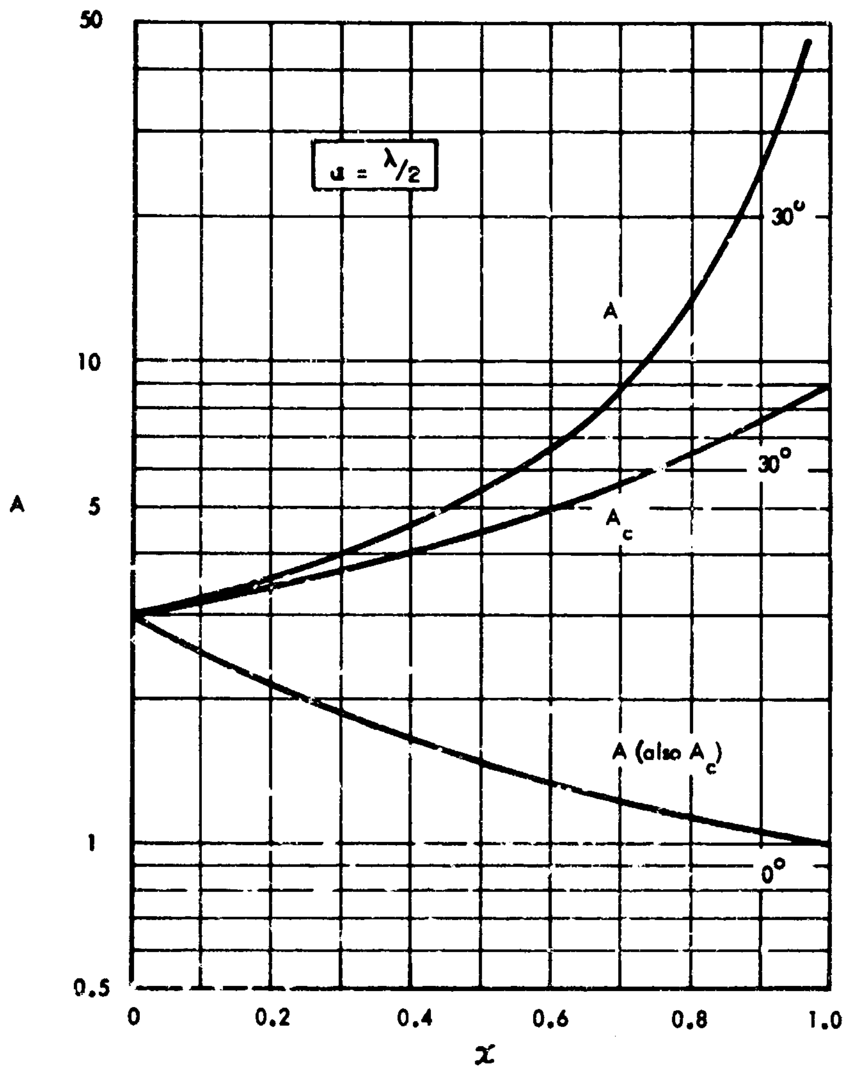

FIGURE 23 ARRAY GAIN VS

VIII. CONCLUSIONS

The primary purpose of this report has been to fit various techniques and criteria related to optimum array processing into a unified theory. Since the models one uses are only approximations to the actual physical situation, it is important to understand how various assumptions affect the optimum receiver structure.

For an interesting class of criteria and signal models we found that the optimum receiver consisted of a set of delays to steer the array followed by a combining operation which depended only on the noise covariance matrix. The output of this combiner is a single waveform, which is then processed depending on the criterion and signal model.

APPENDIX A

DERIVATION OF EQUATION 21

Using Eq. (14) and Eq. (20), we can write:

$$\underline{h}_{\Delta}(t,u) = -\underline{Q}_1(t,u) + \underline{Q}_0(t,u) \tag{A.1}$$

Pre-multiply by $\underline{K}_1(z,t)$, integrate with respect to t, and use Eq. (9):

$$\int_{T_i}^{T_f} \underline{K}_1(z,t)\,\underline{h}(t,u)\,dt = -\underline{I}\,\delta(z-u) + \int_{T_i}^{T_f} \underline{K}_1(z,t)\,\underline{Q}_0(t,u)\,dt \tag{A.2}$$

Post-multiply by $\underline{K}_0(u,x)$, integrate with respect to u, and use Eq. (9):

$$\int_{T_i}^{T_f}\!\!\int \underline{K}_1(z,t)\,\underline{h}_{\Delta}(t,u)\,\underline{K}_0(u,z)\,dt\,du = -\underline{K}_0(z,x) + \underline{K}_1(z,x) \tag{A.3}$$

Observing that

$$\underline{K}_1(z,x) - \underline{K}_0(z,x) = \underline{K}_s(z,x) \tag{A.4}$$

we obtain Eq. (21).

APPENDIX B

EVALUATION OF THRESHOLD

We want to evaluate the sum,

$$\sum_{i=1}^{\infty} \ln\left(1 + \frac{2}{N_0} \lambda_i^{c_1}\right) \tag{B.1}$$

Using Eq. (16), we see that

$$\int_{T_i}^{T_f} \mathrm{Tr}\left[\underline{h}_{c_1}\left(t, t : \frac{2}{N_0}\right)\right] dt = \frac{2}{N_0} \sum_{i=1}^{\infty} \frac{\lambda_i}{1 + \frac{2}{N_0} \lambda_i} \tag{B.2}$$

where the notation $2/N_0$ emphasizes the dependence of $\underline{h}(\cdot)$ on the noise level.

$$\sum_{i=1}^{\infty} \ln\left(1 + \frac{2}{N_0} \lambda_i^{c}\right) = \int_0^z dz \cdot \frac{1}{z} \int_{T_i}^{T_f} \mathrm{Tr}\left[\underline{h}_{c_0}(t, t : z)\right] dt \tag{B.3}$$

One can show that error matrix

$$\underline{\xi}_{c_0}(t, t : z) = \frac{1}{T} \frac{1}{Z} \int_{T_i}^{T_f} \underline{h}_{c_0}(t, t : z)\, dt \tag{B.4}$$

so

$$\ln \Lambda_{b_1}(r(t)) = T \int_0^{2/N_0} \mathrm{Tr}\left[\underline{\xi}_{c_0}(t, t : z)\right] dz \tag{B.5}$$

APPENDIX C

EVALUATION OF d^2 BY USE OF EQUATION 67

In Eq. (104), we derived d^2 using the combined signal. Here, we use Eq. (67) directly.

$$d^2 = \frac{T}{2} \int_{-\infty}^{\infty} \frac{S_s^2(\omega)}{S_n^2(\omega)} \, \mathrm{Tr}\left[\underline{Q}^{-1}(j\omega)\,\underline{P}(j\omega)\,\underline{Q}^{-1}(j\omega)\,\underline{P}(j\omega)\right] \frac{d\omega}{2\pi} \qquad \text{(C.1)}$$

We want to show that

$$\mathrm{Tr}\left[\underline{Q}^{-1}(j\omega)\,\underline{P}(j\omega)\,\underline{Q}^{-1}(j\omega)\,\underline{P}(j\omega)\right] = A^2(\omega)$$

$$= \left[\sum_{i=1}^{N}\sum_{j=1}^{N} \rho^{ij}(j\omega)\right]\left[\sum_{k=1}^{N}\sum_{\ell=1}^{N} \rho^{k\ell}(j\omega)\right] \qquad \text{(C.2)}$$

This is a straightforward exercise in matrix manipulation.

Recall that, after steering,

$$\underline{P}(\omega) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & & & & \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \qquad \text{(C.3)}$$

and

$$\underline{Q}^{-1}(j\omega) = \begin{bmatrix} \rho^{11}(\omega) & \rho^{12}(\omega) & \rho^{13}(\omega) & \rho^{1N}(\omega) \\ \vdots & & \ddots & \vdots \\ \rho^{N1}(\omega) & & & \rho^{NN}(\omega) \end{bmatrix} \qquad \text{(C.4)}$$

Therefore

$$
Q^{-1}(j\omega)\,\underline{P}(j\omega) = \begin{bmatrix} \sum_{j=1}^{N} \rho^{1j}(\omega) & \sum_{j=1}^{N} \rho^{1j}(\omega) & \cdots & \\ \sum_{j=1}^{N} \rho^{2j}(\omega) & \ddots & & \vdots \\ & & & \\ \sum_{j=1}^{N} \rho^{Nj}(\omega) & & & \sum_{j=1}^{N} \rho^{Nj}(\omega) \end{bmatrix} \tag{C.5}
$$

Squaring this, we obtain

$$
\left[Q^{-1}(j\omega)\,\underline{P}(j\omega)\right]^2 = \left[\begin{array}{c|c|c} \sum_{k=1}^{N} \sum_{j=1}^{N} \rho^{1j} \sum_{\ell=1}^{N} \rho^{k\ell} & & \cdots \\ \hline \sum_{k=1}^{N} \sum_{j=1}^{N} \rho^{2j} \sum_{\ell=1}^{N} \rho^{k\ell} & \sum_{k=1}^{N} \sum_{j=1}^{N} \rho^{2j} \sum_{\ell=1}^{N} \rho^{k\ell} & \\ \hline \vdots & & \ddots \end{array}\right] \tag{C.6}
$$

The trace is simply

$$
\sum_{i=1}^{N} \sum_{k=1}^{N} \sum_{j=1}^{N} \rho^{ij} \sum_{\ell=1}^{N} \rho^{k\ell} = A^2(\omega) \tag{C.7}
$$

which is the desired result.

We also observe that

$$\mathrm{Tr}\ [\underline{Q}^{-1}(j\omega)\ \underline{P}(j\omega)] = \sum_{i=1}^{N} \sum_{j=1}^{N} \rho^{ij} \tag{C.8}$$

so that, for this case,

$$\left\{\mathrm{Tr}\ [\underline{Q}^{-1}(j\omega)\ \underline{P}(j\omega)]\right\}^2 = \mathrm{Tr}\left[\left\{\underline{Q}^{-1}(j\omega)\ \underline{P}(j\omega)\right\}^2\right] \tag{C.9}$$

The expression of the right-hand side of Eq. (C.9) is identical to Bryn's result.

REFERENCES

1. W. B. Davenport and W. L. Root, Random Signals and Noise, McGraw-Hill, 1958

2. C. W. Helstrom, Statistical Theory of Signal Detection, London, Pergamon Press, 1960

3. F. B. Hildebrand, Methods of Applied Mathematics, Prentice-Hall, 1952

4. R. Bellman, Introduction to Matrix Analysis, McGraw-Hill, 1960

5. H. L. VanTrees, Detection, Estimation and Modulation Theory, N.Y., Wiley & Sons (to be published in 1967).

6. E. J. Kelly and E. C. Lerner, A Mathematical Model for the Radar Echo from a Random Collection of Scatterers, T.R. No. 123, Lincoln Lab., M.I.T., 15 Jun 1956

7. I. S. Reed, "The Power Spectrum of Echoes from a Random Collection of Moving Scatterers" (paper presented at IDA Summer Study, 1962)

8. H. L. VanTrees, "Optimum Signal Design and Processing for Reverberation - Limited Environments," IEEE Trans. on Mil. Elec., Vol. MIL-9, Nos. 3 and 4, Jul - Oct, 1965, pp. 212-229

9. R. Price, Statistical Theory Applied to Communication Through Multipath Disturbances, M.I.T. Res. Lab. of Electronics, T.R. 266, 3 Sep 1953

10. R. Price, "The Detection of Signals Perturbed by Scatter and Noise," IRE Trans., PGIT-4, Sep 1954, pp. 163-170

11. R. Price, "Optimum Detection of Random Signals in Noise with Application to Scatter-Multipath Communication," IRE Trans., PGIT, Dec 1956, pp. 125-135

12. J. K. Wolf, On the Detection and Estimation Problem for Multiple Non-Stationary Random Processes," Ph.D. Thesis, Dept. of Elec. Engr., Princeton, N.J., 1959

13. F. Bryn, "Optimum Signal Processing of Three-Dimensional Arrays Operating on Gaussian Signals and Noise," J. Acoust. Soc. Amer., Vol. 34, Mar 1962, pp. 289-297

14. D. Middleton and H. Groginsky, "Detection of Random Acoustic Signals by Receivers with Distributed Elements," TR, Raytheon Co., Aug 1963

15. E. J. Kelly and W. L. Root, "A Representation of Vector-Valued Random Processes," J. Math. and Phys., Vol. 39, No. 3, Oct 1960, pp. 211-216

16. H. L. VanTrees, "A Formulation of the Space-Time Processing Problem for Sonar Systems," Project Trident Working Memo. 208, A. D. Little, Inc., 4 Dec 1964

17. R. Price, "Output Signal-to-Noise Ratio as a Criterion in Spread Channel Signalling" M.I.T. Lincoln Laboratory T.R. 388

18. E. J. Kelly, "A Comparison of Seismic Array Processing Schemes," Tech. Note 1965-21, M.I.T. Lincoln Laboratory, 14 Jun 1965

19. S. Darlington, "Linear Least-Squares Smoothing and Prediction, with Applications," BSTJ, Vol. 37, Sep 1958, pp. 1221-1294

20. R.G. Brown and J.W. Nilsson, Introduction to Linear Systems Analysis, J.W. Wiley & Sons, N.Y., 1962

21. M. J. Levin, section in Lincoln Laboratory Semiannual Technical Summary Rept. on Seismic Discrimination, 31 Dec 1964

22. E. J. Kelly and M. J. Levin, Signal Parameter Estimation for Seismometer Arrays, Lincoln Laboratory Tech. Rept. 339, 8 Jan 1964

23. H. W. Marsh, Correlation in Wave Fields, USNUSL Quarterly Rpt., 31 Mar 1950 (C)

24. J. J. Faran and R. Hills, Wide-band Directivity of Receiving Arrays, Harvard Univ. Acoustics Res. Lab., Tech. Memo. No. 31, 1 May 1953

25. R. L. Pritchard, Directivity of Acoustic Linear Point Arrays, Harvard University Acoustics Res. Lab., Tech. Memo. No. 21, 15 Jan 1951

26. A. T. Gaarder, The Design of Point-Detection Arrays, Tech. Rpt., Stanford Research Inst.

27. C. J. Becker and B.F. Cron, Optimum Array Gain for Directional Noise, USL Rpt. No. 656, 5 Oct 1965

GLOSSARY

$A(\omega)$	optimum array gain
$A_c(\omega)$	conventional array gain
$\underline{\alpha}$	unit vector denoting propagation direction
c	velocity of propagation
d_{ij}	inter-element distance
d^2	output signal-to-noise ratio
$\underline{D}(\omega)$	transfer matrix of distortionless combiner
$\delta(\tau)$	delta function
$E[\cdot]$	expectation operation
$E_i[\cdot]$	expectation assuming i^{th} hypothesis is true
η	threshold
F, F_ω	functions used in minimization
$\underline{f}(t)$	known signal
$\underline{g}(\omega)$	transfer function of matrix filter
γ	threshold
$\underline{h}_\Delta, \underline{h}_{c_1}, \underline{h}_{c_0}$	matrix filters
h_i	coefficient in series expansion
H_0, H_1	hypotheses

$\underline{I}$	identity matrix
$\underline{1}$	unity matrix
k	wave number
$\underline{K}_0, \underline{K}_1, \underline{K}_s, \underline{K}_n$	covariance matrices
$\underline{k}(z,t)$	matrix filter
L	sufficient statistic
$\Lambda(\underline{r}(t))$	likelihood ratio
$\Lambda_g(\cdot)$	generalized likelihood ratio
λ	Lagrange multiplier
$\Lambda(\omega)$	filter function
λ_i	scalar eigenvalues
$\underline{m}_1(t)$	mean-value vector on H_1
$\underline{n}(t)$	vector noise process
$n_c(t)$	colored noise component
$n_c(t)$	combined noise component
ω	frequency
$\underline{P}(\omega)$	matrix denoting cross-spectra
$\underline{\varphi}_i(t)$	vector eigenfunction
$\underline{Q}_1(t,u)$	inverse matrix kernel
$\underline{Q}(\omega)$	matrix denoting cross-spectra

$\underline{\rho}(\omega)$	normalized cross-spectral matrix
$\underline{r}(t)$	received vector waveform
$r_i(t)$	components of received vector waveform
$\underline{r}_i$	three-dimensional position vector
$\underline{R}(\omega)$	Fourier transform of $\underline{r}(t)$
$\underline{s}(t)$	vector signal
$s_I(t)$	illuminating signal
$s_R(t)$	reflected signal
$\hat{\underline{s}}(t)$	estimate of vector signal
$s(t:\Omega)$	random signal
$\underline{S}(\omega)$	spectral matrix
$\mathcal{S}(\omega)$	Fourier transform of s(t)
σ_c^2	variance of combined noise
σ^2	variance
ξ_{mmse}	minimum-mean-square error
τ_i	delay
$\tau = t - u$	argument of covariance function
θ_R	phase of reflected signal
θ	angle of directional noise
T_i	initial observation time
T_f	final observation time
T	length of observation interval
μ	$(=\frac{2\pi d}{\lambda})$ normalized variable

V_R	amplitude of reflected signal
w(t)	white noise waveform
y(t)	distortionless output
z(t)	scalar output of combiner

matrix notation:

Tr [ ]	trace
$[\cdot]^T$	transpose
$[\cdot]^{-1}$	inverse
$[\cdot]^*$	conjugate
S^{ij}	ij element in inverse